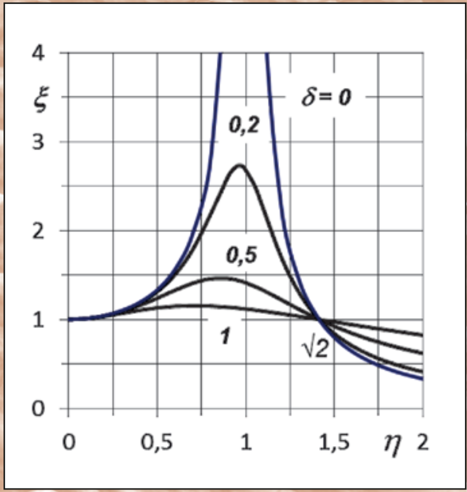
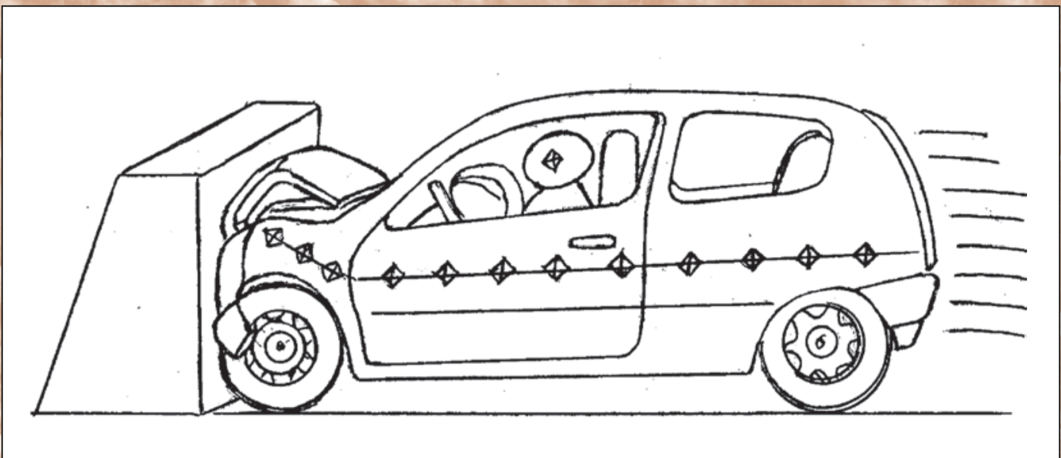


Jaroslav Menčík

Impacts and vibrations

Principles of mechanics and mitigation



University of Pardubice, 2018

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ISBN 978-80-7560-165-0 (print)
ISBN 978-80-7560-166-7 (pdf)

The book is accessible on http://hdl.handle.net/10195/71823 or https://e-shop.upce.cz

Impacts and vibrations

This book deals with the mechanics of impacts and vibrations and mitigation of their effects. It brings the expressions for impacts and for velocity, displacement and force in various cases of braking or stopping, and pays attention to the fact that force impulses propagate in bodies only with limited velocity. Then, it shows how various materials respond to load, how they absorb energy and how they can fail. Further it describes technical means for impact energy absorption, such as bend parts, compressed shells, composites, air cushions and hydraulic shock dampers with constant deceleration. One chapter is devoted to vibrations and mitigation of their effects. Formulae are presented for free and forced vibrations, without or with damping, and attention is also paid to the transmission of forces into foundations and to kinematic exciting, appearing during a vehicle´s ride on a wavy road. Also a dynamic absorber of vibrations is described and vibrations of a system of several bodies. More complex analysis needs numerical methods, such as the finite element method. The last chapter is devoted to the dimensional analysis and theory of similarity, which can spare work during the development of appliances for the damping of impacts and vibrations. The book contains explanatory examples, numerous figures and references.

The book can be downloaded on http://hdl.handle.net/.

Acknowledgment

The author expresses his sincere thanks to both reviewers for careful reading of the manuscript and for revealing various mistakes and drawbacks.

Pardubice, May 2018 Jaroslav Menčík

Rázy a vibrace

Kniha se zabývá mechanikou rázů a vibrací a zmírňováním jejich účinků. Nejprve uvádí základní vztahy pro rázy a dále pro rychlosti, dráhy a síly při různých způsobech zastavování a brzdění. Všímá si i toho, že silové impulsy v tělesech se šíří pouze omezenou rychlostí. Ukazuje, jak se při zatížení chovají různé typy materiálů, jak mohou pohlcovat energii a jak dochází k jejich porušování. V další kapitole se popisují konstrukční prvky pro absorbování energie při nárazech: od ohýbaných dílů přes skořepiny až po kompozity, vzduchové polštáře a hydraulické tlumiče s konstantním zpomalením, jež zajišťuje nejúčinnější brzdění. Samostatná kapitola je věnována vibracím a zmírňování jejich účinků. Jsou uvedeny vztahy pro volné i vynucené kmitání, tlumené i netlumené. Pozornost je věnována i přenosu sil do základů a kinematickému buzení, vyskytujícímu se například při jízdě vozidla po nerovné vozovce. Je také vysvětlen dynamický absorbér vibrací a situace při kmitání více hmot. Ve složitějších případech se pro analýzu užívají numerické postupy, jako je metoda konečných prvků. Poslední kapitola je věnována teorii podobnosti a rozměrové analýze, které mohou ušetřit práci při vývoji zařízení pro tlumení rázů a vibrací. Kniha obsahuje řadu vysvětlujících příkladů, četné obrázky a odkazy na literaturu.

Kniha je volně přístupná na http://hdl.handle.net/.

Poděkování

Je autorovou milou povinností poděkovat upřímně oběma recenzentům této knížky za pečlivé pročtení rukopisu a odhalení různých přehlédnutí a dalších nedostatků.

Pardubice, květen 2018 Jaroslav Menčík

Contents

1. Introduction

Wherever something moves, dynamic forces appear. They are often unfavourable, for example in collisions of vehicles or at impact on an obstacle, during manipulation with various objects, at impact by a hard object of very low mass and high velocity. They appear in transport or technological processes. Higher impact forces can cause damage or even destruction of the bodies. Problems occur also in appliances with rotational or periodic movement, such as motors or machines. If they are not sufficiently balanced, additional forces cause noise and increased wear of bearings, and are a source of parasitic forces and can even lead to an accident, especially at resonance.

Those, who want to face these phenomena and mitigate their consequences efficiently, should know their principal features. Such knowledge brings fruits. From the beginning, cars and other transport means were equipped by suspensions that have been permanently improved. Today, due to the high safety demands, cars and trains use deformable zones, which deform during a collision in a controllable manner and absorb great part of the energy of impact so that the space for the driver and passengers are protected from the consequences of the accident. Well designed and dimensioned machine appliances are safer, with smoother running and longer life. Dynamic absorbers of vibrations, which minimise the vibrations in a special way, namely by adding another vibrating mass, are used in small objects, such as hand shavers, but also in very large structures, such as skyscrapers.

During time, the knowledge of dynamics, including vibrations, has significantly increased. Numerous special literatures exist and also computer programs have been developed, which can solve very complex problems. An important problem therefore is the ability to describe adequately the problem to be solved and to propose the solution and suitable software. This book wants to serve as an introduction to the problems of impacts and vibrations and their mitigation. In addition to the explanation of principal terms and theory for obtaining general idea it gives simple formulae for elementary calculations. Also some technical solutions are mentioned. The individual chapters are complemented by solved problems and references for further study.

The arrangement of the book is as follows. Chapter 2 explains the principles of the mechanics of collisions of bodies: the elastic impact, inelastic and partly elastic. Formulae for velocities and energies are presented, as well as a simplified determination of the maximum force at an impact on an elastic body. Chapter 3 shows the features caused by the limited velocity of propagation of force impulses in bodies and explains the conditions when this must be taken into consideration. Chapter 4 is devoted to the time course of velocity, path of the body, and force for various cases of its braking or slowing down. Attention is also paid to energies. Chapter 5 shows the behaviour of various materials under load: elastic, elastic-plastic and viscoelastic. It explains the distribution of stresses, development of deformations, and energy absorption during elastic-plastic bending. Also it deals with the loss of stability at buckling of slender and thin-walled elements loaded by compressive force, which are often used for mitigation of impacts. It shows the role of stress concentrators and the behaviour of bodies with cracks, and explains the failure of bodies from brittle materials. The following chapter describes structural elements used for energy absorption at impacts: bent rings, compressed shells and other parts, materials with cellular structure such as honeycombs or stiff foams, composites including the types used for ballistic protection, as well as airbags and similar components. It also explains the construction of hydraulic dampers with constant deceleration, which ensure the most efficient stopping and impact damping. Chapter 7 is devoted to vibrations and mitigation of their effects, especially the transmission of forces onto bearings, foundations or constructions in their vicinity. Formulae are presented here for free vibrations without damping and with it and for forced vibrations. Attention is also paid to the kinematic excitation that arises, for example, during a ride of a vehicle on an uneven road. Also a dynamic absorber of vibrations is described, which uses a mass attached via a spring to the vibrating body. With proper tuning it is only this additional mass that vibrates, while the main body, which is excited, remains calm. The vibrations of a system of several bodies are also mentioned briefly. More complex cases of impact load or vibrations are solved by numerical procedures, such as the finite element method. Such solutions need suitable computer programs, which are also mentioned briefly. The last chapter is devoted to the dimensional analysis and theory of similarity, which enable reduction of the extent of computations and experiments and generalisation of the results and thus savings in the research and development of various appliances for damping and mitigation of impacts and vibrations.

2. Impacts and hits

2.1 Introduction

Impacts of two or more bodies occur very often. Examples are: a collision of two bodies of comparable mass, an impact of a body on a strong barrier, or a strike by a body of high or very small mass. They often occur in transport and also in various technological processes. Practical examples are:

- Collisions of transport means (cars, trains, ships, airplanes) or impacts of them on various structures or barriers.
- Fall of objects from height (e.g. a vehicle, fall of an object during manipulation, fall of stones on the road, a vehicle or transport belt).
- Strike by a flying object (a projectile, hail, stream of sand or small balls in shot-peening for surface strengthening or for cleaning of various parts).
- Strike by a hammer or power-hammer in forging, work of a pneumatic pick or a percussion drill.
- Driving of nails into wood or piles into the soil.
- Work of machines or mechanisms with reversible or repeated movements (looms, manipulation with moulds in foundries or in glass works, conveyers, manipulators and robots, planers, crane bridges, valves in combustion engines, doors in transport means, and many others).

Often it is also necessary to stop a movement within a very short distance. For example a train whose brakes failed, must not leave the rails at the end of platform.

There are also examples from the world of entertainment: collisions of billiard balls or return of a ball in tennis or football. Some readers may remember an artist performance in a circus arena, where one man laid on the ground with a large stone on his breast, and another man hit the stone by a big hammer. Most spectators were horrified at this instant. But those, who had known principles of mechanics, sat calm, as they knew that nothing could happen.

A special case is so-called hydraulic impact that occurs if the flow of liquid is stopped suddenly. (This phenomenon will be explained at the end of the next chapter.)

In all these examples small or high forces arise. Sometimes they can cause destruction of the bodies in contact or increased noise and wear. The task of this chapter is to provide a general idea so that the reader can orientate himself in solution of these problems with the aim to minimise the forces and damage, or – on the contrary – to utilise efficiently the energy of an impact. Here, we shall remind the principal concepts and relationships of the pertinent branches of mechanics and illustrate their applications on simple problems. More comprehensive materials can be found in textbooks of dynamics and other literature, for example [1 – 6].

The base for solution of impact problems are Newton's laws and equations of motion, the laws of conservation of momentum and energy, and expressions for the calculation of kinetic energy and work of deformation. The problem is very simple if only the velocities of bodies after collision must be determined. Here, we shall look at velocities, deceleration, braking distances and duration of impact in several basic cases. The forces at impact will be considered here only very briefly, as they will be treated in more detail in a separate chapter.

2.2 Direct collision of mass points

The situation can be illustrated on two elastic balls of masses m_1, m_2, which move in horizontal direction along the same straight line (Fig. 2.1, left). The velocity of the first ball is higher than that of the second ball, $v_{10} > v_{20}$, and they touch one another. Since this instant force N acts among them, both balls are compressed and the distance between their centres of gravity gets smaller. This is the *compression period*, which ends at time t_c, when this distance is minimum and the contact force is the highest (Fig. 2.1, centre). At this instant the velocity of both bodies equals v_c. Then, the *stage of restitution* follows. If the deformations were only elastic, a part of the kinetic energy was changed into potential energy of elastic stresses. This energy is gradually released during the restitution period, the deformations become smaller and after certain time both bodies get apart and continue in their movement

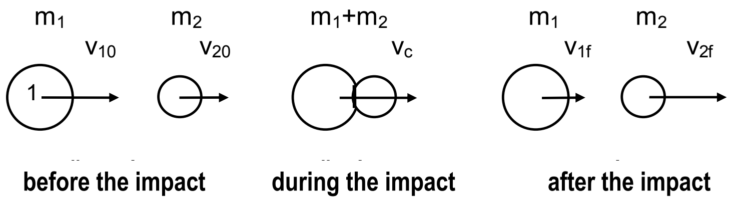

Figure 2.1. *Collision of two bodies*

by different velocities v_{1f}, v_{2f} (Fig. 2.1, right).

This is *elastic impact*. Sometimes, no springing back occurs and both bodies continue as a whole by velocity v_c. This is *inelastic impact*. Usually, partial rebound occurs, and one speaks about *partially elastic impact*. Now, we shall look at the individual cases. We start with the velocity v_c at the end of the first period, which is common to all cases. This velocity can be obtained from the law of momentum conservation. It is

$$m_1 v_{10} + m_2 v_{20} = (m_1 + m_2)\, v_c \,, \tag{2.1}$$

from which it follows

$$v_c = (m_1 v_{10} + m_2 v_{20}) \,/\, (m_1 + m_2) \,. \tag{2.2}$$

2.3 Elastic impact

If the impact was elastic, no energy was consumed (dissipated) during it. The velocities after springing back can be thus determined via the law of **energy conservation**, which says that the sum of kinetic energies after the impact must be the same as before the impact:

$$E_{kin,0} = E_{kin,f} = ½\, m_1 v_{10}^2 + ½\, m_2 v_{20}^2 = ½\, m_1 v_{1f}^2 + ½\, m_2 v_{2f}^2 \,. \tag{2.3}$$

Also the law of **momentum conservation** of this system of two bodies is valid:

$$m_1 v_{10} + m_2 v_{20} = m_1 v_{1f} + m_2 v_{2f} \tag{2.4}$$

After expressing the velocity v_{2f} from this equation and inserting it into Equation (2.3) one obtains, after a rearrangement, the final velocity of body 1:

$$v_{1f} = \frac{(m_1 - m_2)v_1 + 2m_2 v_2}{m_1 + m_2}, \tag{2.5}$$

Similar procedure gives the final velocity of the second body:

$$v_{2f} = \frac{(m_2 - m_1)v_2 + 2m_1 v_1}{m_1 + m_2} \,. \tag{2.6}$$

Let us look at some examples. The reader is encouraged to solve them also, with the use of formulae (2.2), (2.5) a (2.6).

<u>Case A</u>. Both balls have the same mass ($m_1 = m_2$) and move one against the other by the same velocities ($v_{20} = -\, v_{10}$; the minus sign means that the direction v_{20} is

opposite to direction v_{10}). Inserting these values into Equations (2.2), (2.5) and (2.6) gives $v_c = 0$, $v_{1f} = -v_{10}$ a $v_{2f} = -v_{20} = v_{10}$. The balls bounce back and each moves by its original velocity in the opposite direction.

Case B. Both balls have the same mass, the ball 1 moves by velocity v_{10}, the ball 2 does not move ($v_{20} = 0$). It follows from Eqs. (2.2), (2.5) and (2.6) that $v_c = ½\, v_{10}$, $v_{1f} = 0$, $v_{2f} = v_{10}$. The first ball stops, while the other starts moving by the velocity v_{10} in the direction of initial movement of the first ball. The ball 1 has passed its momentum on the ball 2. Both balls thus have exchanged their velocities.

Case C. The second ball has much higher mass than the first one, $m_2 >> m_1$, and does not move ($v_{20} = 0$). The common velocity at the end of the compression period is therefore $v_c = 0$. The velocities after the collision are: $v_{1f} = -\, v_{10}$, $v_{2f} = 0$. The second ball does not move (it is much larger than the first ball), and ball 1 jumps back with the same velocity with which it fell on the second ball.

Case D. The second ball has much higher mass than the first ball and the same velocity, but in the opposite direction, $m_2 >> m_1$, $v_{20} = -\, v_{10}$. The common velocity at the end of the compression period is $v_c = -\, v_{10}$, and the velocities after the collision will be: $v_{1f} = -\, 3v_{10}$, $v_{2f} = -\, v_{10}$. The second ball practically does not change its velocity (it is much larger than the first ball), but the ball No. 1 jumps back by higher velocity than the velocity of its impact on the second ball. Something similar happens if a car going on a road loses a wheel, which continues moving (by inertia) in its original direction and hits a much heavier car coming from the opposite direction. The wheel is flung back by very high velocity and can hit another car and damage it.

In all cases the difference of velocities of both material points after the collision equals the opposite difference of their velocities before the collision. This holds for elastic impact in general. One can also say that the absolute value of the difference of the velocities of both elastic bodies after the impact is the same as before it.

Despite of significant simplifications, the presented examples show clearly that in a head-on collision of two cars the lighter vehicles is at a disadvantage, especially if the other vehicle is much heavier. A light vehicle can be damaged and flung away and the collision can be fatal for the passengers, while the driver of much heavier vehicle will feel only slight impact and its velocity changes only a little.

Ideally elastic impact is one extreme case. Now we shall look at the other extreme.

2.4 Inelastic impact

If the impact were perfectly inelastic, if, for example, both bodies were from modelling clay and join during the collision strongly so that they form one body, they would finally move by the velocity v_c, defined by Equation (2.2). Kinetic energy of this system will be

$$E_{kin,f} = ½\,(m_1+m_2)\,v_c^2 \; . \tag{2.7}$$

It is smaller than the initial energy $E_{kin,0}$ in Eq. (2.3), because a part of it was changed into deformation work (dissipation). The loss of energy is

$$\Delta E_{kin} = ½\,m_1v_{10}^2 + ½\,m_2v_{20}^2 - ½\,(m_1+m_2)v_c^2 = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left(v_{10}-v_{20}\right)^2 . \tag{2.8}$$

In reality, however, no impact is perfectly elastic or inelastic, and can be denoted as partly elastic.

2.5 Partly elastic impact

During the first stage of impact a part of the kinetic energy is accumulated in the bodies as potential energy of elastic stresses (E_{el}), and a part of the energy (W) was consumed for plastic deforming of one or both bodies, for the creation of fracture surfaces when damaged bodies break, and also in friction and other processes, among other for the generation of noise during the collision. W denotes the work done at the impact, or, generally, dissipated energy. Only elastic energy is released during the rebounding. The momentum conservation law (2.4) now holds similarly to elastic impact, but the law of energy conservation (2.3) changes to

$$½\,m_1v_{10}^2 + ½\,m_2v_{20}^2 = ½\,m_1v_{1f}^2 + ½\,m_2v_{2f}^2 + W\,. \tag{2.9}$$

The velocities after impact can be obtained by solving Eqs. (2.4) and (2.9). Equation (2.4) is squared, the terms ($½\,m_1v_{1f}^2$ + $½\,m_2v_{2f}^2$) in equation (2.9) are transferred to the left side, then this equation is multiplied by [$-2(m_1 + m_2)$] and both expressions are summed. In this way we obtain

$$(v_{1f}-v_{2f})^2 = (v_{10}-v_{20})^2 - 2\frac{m_1+m_2}{m_1\,m_2}W \; . \tag{2.10}$$

Expressing v_{2f} from Eq. (2.4) and rearranging, we get the final velocity of body 1:

$$v_{1f} = v_c - \frac{m_2}{m_1 + m_2}\sqrt{(v_{10} - v_{20})^2 - 2\,\frac{m_1 + m_2}{m_1\, m_2}W} \;; \tag{2.11}$$

v_c is the common velocity of both bodies at the end of compression stage, defined by Equation (2.2). Similarly we obtain the final velocity of the second body:

$$v_{2f} = v_c + \frac{m_1}{m_1 + m_2}\sqrt{(v_{10} - v_{20})^2 - 2\,\frac{m_1 + m_2}{m_1\, m_2}W} \;. \tag{2.12}$$

These expressions hold also for the velocities during the impact. In this case W expresses the work consumed till the investigated instant. For $W = 0$, that is no consumed energy during the impact, Equations (2.11) and (2.12) change to equations (2.5) a (2.6) for elastic impact.

For simpler evaluation of impacts with various bodies Newton has defined **coefficient of restitution ε** as the ratio of the difference of velocities of both bodies after the impact and the difference of the velocities before the impact,

$$\varepsilon = \left| \frac{v_{1f} - v_{2f}}{v_{10} - v_{20}} \right| . \tag{2.13}$$

The coefficient of restitution for an elastic impact is $\varepsilon = 1$, while for inelastic one it is $\varepsilon = 0$. For partly elastic impact ε varies between 0 and 1; it is the closer to 1, the bigger rebound.

The coefficient of restitution can be determined in experimental way, for example from the height h_f of the rebound of a ball from the investigated material after its fall on a massive body from height h_0. In this case, h_0 and h_f are known, and also the velocity of impact $v_{10} = \sqrt{(2gh_0)}$ and rebound velocity $v_{1f} = \sqrt{(2gh_f)}$; for massive base $v_{20} = v_{2f} = 0$. After inserting these values into Eq. (2.13) one obtains

$$\varepsilon = \sqrt{\frac{h_f}{h_0}} \;. \tag{2.14}$$

The potential energy in gravitational field is

$$U_{pot} = m\, g\, h \;; \tag{2.15}$$

g is the acceleration of gravity, m is the mass of the body and h is the height of its fall. Expression (2.14) also says that the coefficient of restitution equals the square root of the ratio of the potential energies of the body after the impact and before it.

REMARK. In the past, the coefficient of restitution was considered as material constant. Table 1 shows values for several material pairs. The situation, however, is more complex, as the behaviour at impact is influenced also by the condition of surface of both bodies (e.g. roughness) and also by the velocity of impact and other factors. Table 1 can thus serve only for orientation.

Table 1. Coefficients of restitution of various materials *[5, 7]*

material (pair of materials)	body	ε
quenched steel	*bearing balls*	*0.98*
glass – glass	*ball and massive plate*	*0.94*
structural steel	*ball and massive plate*	*0.93*
ivory	*billiard balls*	*0.82*
rubber – marble	*rubber ball, massive plate*	*0.82*
cast iron	*balls*	*0.68*
wood (elm)	*balls*	*0.60*
bell bronze	*balls*	*0.59*
lead	*balls*	*0.20*
clay	*balls*	*0.17*

Now we shall look at the relation between the coefficient of restitution and the energy consumed during the impact. It follows from Eq. (2.9) that the consumed energy equals the difference of energies of both bodies before and after the impact,

$$W = \tfrac{1}{2}\, m_{10}^2 + \tfrac{1}{2}\, m_{20}^2 - \tfrac{1}{2}\, m_{1f}^2 - \tfrac{1}{2}\, m_{2f}^2 \,, \tag{2.16}$$

Expression of the velocities in Eq. (2.13) by means of Equations (2.11) a (2.12) and a rearrangement gives

$$\varepsilon = \frac{\sqrt{(v_{10} - v_{20})^2 - 2\,\dfrac{m_1 + m_2}{m_1\, m_2}\, W}}{v_{10} - v_{20}} \,, \tag{2.17}$$

$$W = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_{10} - v_{20})^2 (1 - \varepsilon^2) \quad . \tag{2.18}$$

For $v_{20} = 0$ and $m_2 >> m_1$ (that is, for impact of body 1 on a massive body 2) one gets the following simple relationship between the coefficient of restitution and the ratio of the work dissipated at impact and the initial kinetic energy:

$$\varepsilon = \sqrt{1 - (W / E_{1,kin,0})} \text{ , and also } W / E_{1,\mathrm{kin},0} = 1 - \varepsilon^2 \, . \tag{2.19}$$

With the known coefficient of restitution ε and the velocity v_c at the end of compression, given by Eq. (2.2), the formulae for final velocities can be written as:

$$v_{1f} = v_c - \frac{m_2}{m_1 + m_2} (v_{10} - v_{20}) \varepsilon \ , \tag{2.20}$$

$$v_{2f} = v_c + \frac{m_1}{m_1 + m_2} (v_{10} - v_{20}) \varepsilon \ . \tag{2.21}$$

They differ from expressions (2.5) and (2.6) only by the coefficient of restitution ε.

2.6 Oblique impact

The situation in real impacts usually differs from the ideal case of impact of two balls moving on the same line. For example, so-called *oblique impact* occurs if the vectors of velocity of both bodies do not lie on the same line, but intersect. In such case the velocities and forces at the contact have general directions, which have a component in the normal direction to the contact surface and a component in the direction of the tangent to this surface. For the velocity components in the normal direction everything said above holds. The tangential components remain without any change. Similar situation also exists in the case when the vectors of velocities of both bodies lie on the same line, but the plane, tangent to the surfaces of both bodies, is not perpendicular to this line. The effect of contact forces in normal direction will be addressed in the following chapter. The tangential component of the force acts at some distance from the centroids and tries to rotate each body, so that the residual movement can be much more complex. More to this topic can be found, for example, in [1].

The use of the derived expressions will be illustrated on three practical problems.

Problem 1. Efficiency of forging

A hammer of mass m_1 falls by velocity v_{10} on a forged object lying on an anvil, whose common mass is m_2 and velocity $v_{20} = 0$ (Fig 2.2a). We shall assume that the impact is perfectly inelastic so that the hammer does not rebound.

If the efficiency of forging should be the highest, most energy should be passed on the forged object. It means that the loss of energy of the hammer should be the highest. Expression (2.8) thus changes for $v_{20} = 0$ to

$$\Delta E_{kin} = \frac{1}{2} m_1 v_{10}^{\ 2} \frac{m_2}{m_1 + m_2} = E_{kin,1,0} \frac{m_2}{m_1 + m_2} \quad . \tag{2.22}$$

This value is maximum if the fraction $m_2/(m_1 + m_2)$ is the highest. This will be for $m_2 >> m_1$. The most efficient forging is achieved if the mass of the forged body and the anvil is biggest compared to the mass of the hammer.

A very similar case is the impact of a hammer on a stone lying on the breast of a circus artist. Big mass of the stone significantly damps the effect of the stroke.

Problem 2. Efficiency of the pile driving into soil

Some structures on soils of low load-carrying capacity rest on piles driven into the soil. During driving-in a hammer of mass m_1 falls on the pile and passes on it a part of its kinetic energy (Fig. 2.2b). This impact is partly elastic and due to transmitted energy the pile penetrates into the soil. The amount of energy passed on the pile is the highest if the loss of kinetic energy of the hammer is the smallest. Contrariwise to forging the formula (2.22) must have the smallest value. This is achieved if the mass of the pile driver is highest compared to the mass of the pile, $m_1 >> m_2$.

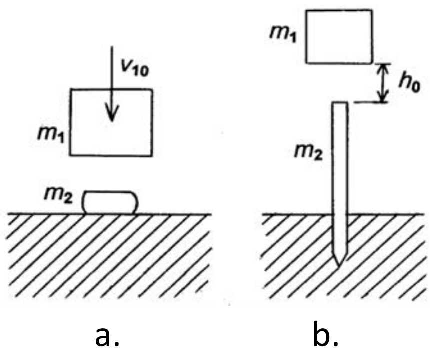

Figure 2.2. *Forging (a) and pile driving (b)*

REMARK. h_0 is the fall height of the pile driver, considered usually as the distance of the pile driver and pile before the impact. In fact, also potential energy of the hammer is released during its movement during the pile penetration into the soil. This component is usually small compared to the total height h_0 of the fall.

Problem 3. Determination of velocity of a bullet by means of ballistic pendulum

The schematic is in Figure 2.3. The projectile penetrates into the pendulum body (e.g. a case with sand) and causes it to swing. The bullet velocity is then determined from the swing height. The problem will be solved for the following values: bullet mass $m_1 = 0.01$ kg, pendulum mass $m_2 = 10$ kg, the pendulum arm $R = 3$ m, height of swing of the pendulum centroid $h = 5$ mm.

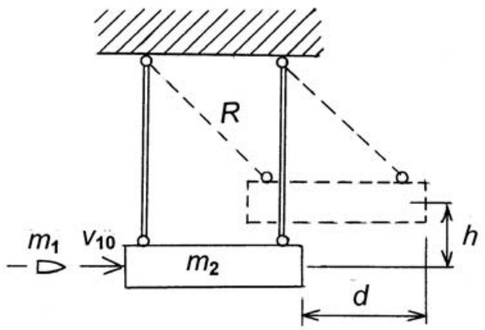

Figure 2.3. *Ballistic pendulum*

Solution. The bullet penetrates into the pendulum and passes its energy on it. This impact can be considered as fully inelastic. A part of the kinetic energy of the projectile is dissipated by its plastic deformation and by mutual friction and crushing of sand grains, and another part is changed into kinetic energy of the pendulum and, gradually, into its potential energy. The amount of dissipated energy is not known yet. The energy conservation law "kinetic energy of the projectile changes into potential energy of the pendulum" thus cannot be used. The solution will be based on the law of momentum conservation: "the initial momentum of the bullet equals the momentum of the system "bullet + pendulum" at the first instant of bullet penetration",

$$m_1 v_{10} = (m_1 + m_2) v_c ; \qquad (2.24)$$

v_c is the pendulum velocity immediately after the penetration of the bullet. At this instant the system "pendulum plus bullet" has kinetic energy $E_{kin} = ½ (m_1 + m_2) v_c^2$, which changes into the potential energy of the system, $E_{pot} = (m_1 + m_2)gh$, where g is acceleration of gravity (9.81 m/s^2) and h is the distance between the height of the centroid of the system at the swing and the initial height. The condition of equality of both energies gives (after a rearrangement)

$$v_{1,0} = \frac{m_1 + m_2}{m_1}\sqrt{2gh} \tag{2.25}$$

The initial velocity of the bullet for the above values is v_{10} = 313.5 m/s.

Comment 1. We have assumed that the bullet penetrated into the pendulum at its centroid and that the pendulum could not rotate, so that its moment of inertia was not considered. As very small heights of swing are difficult to measure, one usually measures the horizontal displacement of the pendulum, $d = \sqrt{[R^2 - (R - h)^2]}$, which, in our case is d = 173 mm. Now, it is necessary to check, whether the calculated time t_b for stopping the bullet was significantly shorter than the period of pendulum oscillation, $T = 2\pi\sqrt{(R/g)}$. The movement of the bullet in the pendulum can be approximately considered as uniformly delayed, so that the time of stopping at the distance d is $t_b = 2d/v_{10} = 2\times0.173/300 = 0.001153$ s. This is much less than $T =$ $= 2\pi\sqrt{(3.0/9.81)} = 3.475$ s, so that no further correction is necessary.

Comment 2. Kinetic energy of the bullet before penetrating into the pendulum was $E_{kin} = ½\, m_1 v_{10}^2 = 450.0$ J. The potential energy of the pendulum with the bullet after the swing is $E_{pot} = (m_1+m_2)gh = (0.01+10)\times9.81\times0.005 = 0.491$ J. This is only a little more than one thousandth of the initial energy (449.509 J). This means that nearly 99.9 % was absorbed by irreversible processes. If this energy would not be absorbed, the pendulum would swing as high as 4.58 m (provided its construction would allow it), or the bullet would cause the corresponding damage. This illustrates the great importance of energy dissipation for the mitigation of impact consequences.

2.7 Forces at impacts – simplified solution

Essential for the determination of forces is the knowledge of the relationship between the force and deformation. The contact details will be investigated later. This section will be limited on the strike of a rigid body of mass m and velocity v_0 on a spring (of negligible mass) attached to a rigid base. The word "spring" can denote any elastic body.

Since the spring is touched by the body, it is compressed by the force

$$F = k\,x\,; \tag{2.26}$$

k is the spring stiffness and x its compression, equal to the body displacement.

The problem could be solved by the methods of dynamics using the basic equation $F = ma$, which would give the time course of the displacement and force. This will be shown in Chapter 5. If only the maximum force should be known, we can get it from the law of energy conservation. The moving body has kinetic energy

$$E_{\text{kin}} = \tfrac{1}{2}\, m v_0^2\,. \tag{2.27}$$

During the spring compression, the kinetic energy of the body changes into deformation energy of the spring. This energy can be expressed (see Eq. 2.26) as

$$E_{\text{pot}} = \tfrac{1}{2}\, Fx = \tfrac{1}{2}\, F^2/k = \tfrac{1}{2}\, kx^2 \tag{2.28}$$

At the instant of maximum compression the body is at rest and its kinetic energy was changed into the potential energy of the spring,

$$E_{\text{kin}} = E_{\text{pot}} = \tfrac{1}{2}\, F^2/k\,. \tag{2.29}$$

Combination of Eqs. (2.27) and (2.28) gives the maximum force

$$F_{\text{max}} = \sqrt{2W_{kin}k} = v_0\sqrt{mk}\,. \tag{2.30}$$

We can see that the maximum force at the impact is directly proportional to the velocity of impact $\boldsymbol{v_0}$ and is higher for higher stiffness of the spring or bumper and higher mass of the moving body; in both cases it grows with their square root.

Also the knowledge of maximum deceleration is important. At an impact of a car on an obstacle a question is what the consequences are for the passengers or cargo, and about the strength of their fixing. The basic dynamic equation,

$$F = m\,a\,, \tag{2.31}$$

gives that the maximum deceleration will be

$$a_{\text{max}} = F_{\text{max}}/m = v_0\sqrt{k/m} \tag{2.32}$$

REMARK. All cases here considered horizontal movement. If the bodies move in vertical direction, the changes of their potential energy in gravitation field must be considered, as well. A fall from the height h releases energy $E_{\text{pol}} = mgh$. This energy must be added to the kinetic energy of the moving body. Generally, one should work with the position of the centroid of body 1 during compression of bodies 1 and 2. This fact can be important especially for bodies of high compliance.

The importance of spring compliance will be illustrated on the following problem.

Example.

A vehicle with elastic bumper, going by velocity v = 10 km/h, hits a massive stiff wall. What is the maximum force at the impact, the time to stop and maximum deceleration? Assume the mass of the vehicle m = 1000 kg and stiffness of the bumper k = 500 N/mm = 500 kN/m.

Solution. The maximum force at impact can easily be obtained from the law of energy conservation. At the beginning the vehicle has kinetic energy, which will be converted to the potential energy of the bumper spring, $E_{kin} = U_{pot}$. The velocity is v = 10000/3600 = 2.78 m/s and kinetic energy E_{kin} = 3865 J. Equation (2.30) yields

$$F_{max} = v\sqrt{mk} = 2{,}78\sqrt{1000 \times 500000} = 62163 \text{ N} .$$

The path to stopping, corresponding to this force, is $x_b = F_{max}/k$ = 62163/500000 = 0,1243 m. The maximum deceleration can be obtained from the basic equation for movement: $F = ma$. From here, $a_{max} = F_{max}/m$ = 62163/1000 = 62.16 m/s^2. This is more than 6 g ! The time to stopping will be t_b = 0.070 s; the pertinent formula (4.18) is given in Chapter 4.

This impact was at relatively slow velocity, 10 km/h. The force and deceleration at the speed 50 km/h would be 5-times higher (i.e. 30g !); cf. Eqs. (2.30) and (2.32).

The energy of impact is proportional to the square of velocity; it is thus 25-times higher. This had to be considered in the design of the bumper or energy absorber.

Let us look at the situation of a spring with ten times higher compliance, so that the stiffness is $k' = k/10$ = 50000 N/m. The impact velocity is again 10 km/h. The corresponding maximal force will be $F_{max} = 2.78\sqrt{(1000\times50000)}$ = 19657.6 N, i.e. 3.16-times lower. Maximum deceleration, which is always at the end of stopping for a linear spring, will be a_{max} = 19657.6/1000 = 19.66 m/s^2, i.e. about 2g. The braking distance will be x_b = 19657.6/50000 = 0,3932 m = 39.3 cm. The time to stopping is 0.222 s. One can say roughly that ten-times increase of the spring compliance has led to three-times lower maximum force and deceleration, and three times longer time to stopping and braking distance. (And thus three times larger deformation of the bumping element).

REMARK. In this example, impact of a rigid body with an elastic bumper was considered. Often also the moving body, e.g. a car, deforms at the impact. This reduces the effects of the impact, as, from the mechanics point of view; both bodies are arranged in series (see Chapter 4.2, sub-chapter Several springs in series). However, the use of purely elastic elements for the stopping of a moving body would have two drawbacks. In comparison with the element causing constant deceleration, the maximum force corresponding to a simple spring will be at least twice as high (see Chapter 4). Moreover, due to the accumulation of energy in the bumper, the stopped body will have a tendency to spring back. More appropriate are therefore elements that absorb or dissipate the energy, for example by plastic deforming. Some possibilities will be shown in Chapter 6.

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3. Wave character of stress increase

Any force impulse can propagate only with limited velocity. This fact is usually neglected, and the forces and deformations are calculated according to the theory based on the relationships between stress and strain under static load. In collisions of bodies and also in other cases of impact loading it is sometimes necessary to account for the finite velocity of propagation of force impulses in elastic materials. At the beginning all particles of a body are at rest and deformations, caused by the force, propagate from the place of disturbance in the form of elastic waves. The velocity of their propagation is high, but if the dimensions of the body are large, the duration of passage of waves through the body increases, and this phenomenon should be taken into account [1 – 3]. The situation can be illustrated on a simple problem where a one end of a long bar from elastic material is hit by a massive rigid body moving in the direction of the bar axis (Fig. 3.1). At the end of this chapter we also shall look at the hydraulic impact in liquids.

3.1 Longitudinal waves in a prismatic bar

We shall assume that the bar is from an isotropic material of density ρ and modulus of elasticity E, and has the length L and cross section area S. The left end of the bar is free and the right end rests on a rigid wall. For simplicity, we shall consider only the forces, movement and deformations in axial direction, but we shall not assume that the compressive force could cause buckling or deflection of the bar. The moving body has velocity v and the mass m much bigger than the bar.

The situation is depicted in the right part of Figure 3.1. Using two fictitious cuts at the distance x from the hit end we take out an element of infinitesimal thickness dx from the bar and write the equation of equilibrium of forces acting on it. Due to these forces, the left section moves by u and the right one moves by u + du, or by $u + (\partial u / \partial x)\,\mathrm{d}x$. The investigated element thus becomes longer by $(\partial u / \partial x)\,\mathrm{d}x$.

NOTE. As these displacements depend simultaneously on the position x and time t, partial derivatives and symbol ∂ are here used instead of simple derivatives (symbol d), used for functions of one variable.

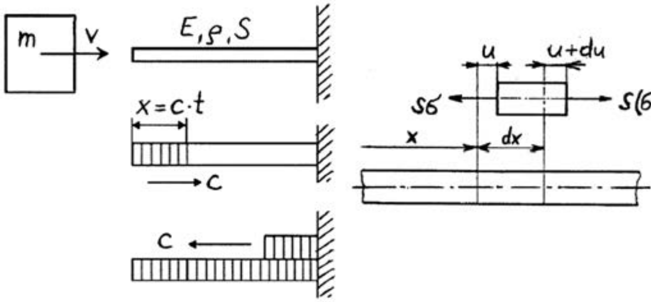

Figure 3.1 *Left: Impact of a massive body on a bar, and propagation of stress waves (below). Right: the forces acting on an element of the bar.*

The **strain** (**relative elongation**) is $\varepsilon = -\,\partial u/\partial x$ and the corresponding stress equals $\sigma = E\varepsilon$, where E is modulus of elasticity of the bar (Young modulus).

Generally, the stress and deformation vary continuously along the bar. In the left cross section of the element the stress $\sigma(x)$ acts, while stress $\sigma(x+\mathrm{d}x) = \sigma(x) + \mathrm{d}\sigma$ acts in the right section. The corresponding forces, equal to the product of stress and area of the cross section, are $F(x)$ and $F(x) + \mathrm{d}F$. The difference of both forces, $\mathrm{d}F = S\,\mathrm{d}\sigma\,[=SE\mathrm{d}\varepsilon = SE(\partial u/\partial x)\mathrm{d}x]$, gives this element the acceleration $\ddot{u} = \partial^2 u/\partial t^2 = \mathrm{d}F/\mathrm{d}m$, where $\mathrm{d}m$ is its mass, for which it holds $\mathrm{d}m = \rho\,S\,\mathrm{d}x$; ρ being the density of the bar. These quantities can now be used to write the equilibrium of forces acting on the infinitesimal element of the bar:

$$S\,\mathrm{d}\sigma = \mathrm{d}m\,\frac{\partial^2 u}{\partial t^2} = \rho\,S\,\mathrm{d}x\,\frac{\partial^2 u}{\partial t^2} = SE\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\,. \tag{3.1}$$

A rearrangement gives the basic equation for the propagation of stress waves in the direction of the bar axis:

$$\partial^2 u/\partial t^2 = c^2\,(\partial^2 u/\partial x^2)\,, \tag{3.2}$$

where

$$c = \sqrt{\frac{E}{\rho}} \tag{3.3}$$

is the velocity of propagation of longitudinal elastic waves in one-dimensional medium, so-called phase velocity. For steel (E = 210 GPa, ρ = 7850 kg/m^3): c = 5172 m/s, for concrete (E = 44.0 GPa, ρ = 1950 kg/m^3): c = 4750 m/s. The velocity v (= $\partial u/\partial t$) of the particle movement, following from the solution of Equation (3.2), is $v = - c(\partial u/\partial x)$.

If a massive rigid body hits the left end of the bar by velocity v_0, compressive stress

$$\sigma_0 = E\frac{v_0}{c} = v_0\sqrt{E\rho} \tag{3.4}$$

starts propagating from here by velocity c. At time t the elastic wave arrives at the distance $x = ct$. In the influenced region compressive stress σ_0 acts; right of it no stress acts yet (Fig. 3.1, left). The compressive stress causes the shortening of the loaded part of the bar by $\Delta x = x\sigma_0/E = v_0x/c$. This shortening is identical with the displacement of the moving body during the time t, which makes $u = v_0t = v_0x/c$.

Further, we shall assume that the stress is distributed uniformly in the cross section, and is lower than the yield strength of the bar material. (The case of higher stress will be looked on later.) The region with σ_0 grows by the velocity c till the instant when the front of the stress wave arrives at the rigid wall. This is in time

$$\Delta t = L/c\ . \tag{3.5}$$

The compressive stress wave is reflected back again as compressive (of the same magnitude) and propagates leftwards (Fig. 3.1 left down). The resultant stress in the influenced region equals the sum of the initial and reflected wave, that is $\sigma_2 = 2\sigma_0$. The body, whose mass m is very big, moves further with the same velocity. When the reflected wave reaches at time $2\Delta t$ the left end of the bar, it again reflects from the massive body and moves as compressive wave rightwards. The resultant stress in the influenced region is $\sigma_3 = 3\sigma_0$. The process continues in this way further, with the stress increasing by steps $2\sigma_0$ in time intervals $2\Delta t$ (Fig. 3.2).

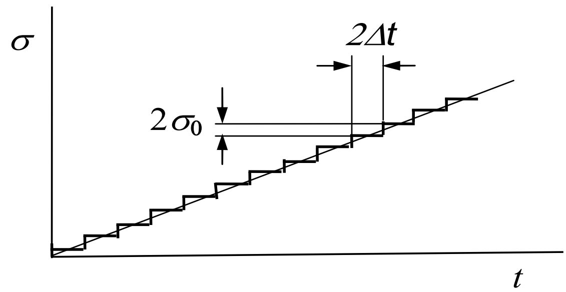

***Figure 3.2**. Gradual growth of stress at the left end of the bar.*

For an idea: the time needed by an impulse for passing a steel bar of length 1 m is Δt = 0.000193 s. Back to the left end it arrives at $2\Delta t$ = 0.000387 s. These are very short times, and for small bodies with characteristic dimensions of the order of centimetres, they are much shorter.

If the right end of the bar is free, the stress wave is also reflected here, but not as compressive, but tensile. This has one unexpected consequence. If the bar is made of a brittle material and the stress exceeds its tensile strength, a small part of the bar near its free end tears off and flies away. Similar behaviour can be observed with Newton´s cradle [4, 5], depicted in Fig. 3.3. If the ball at the end of a series of balls hanging on thin fibres is moved aside and released, it swings back and hits the neighbouring ball and passes its impulse on it. This impulse propagates (invisibly) through the series and when it arrives at the last ball at right, this ball jumps away and falls back. This new impulse propagates leftwards to the first ball, which jumps away a little less, etc. After some time the process ends due to energy losses.

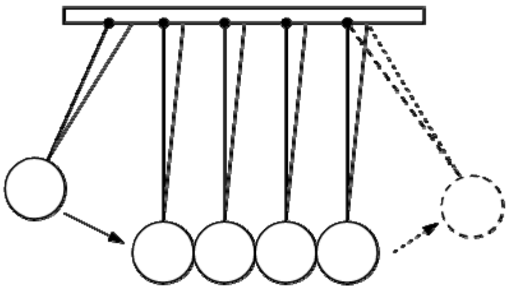

Figure 3.3. *Newton´s cradle*

Let us return to our long bar hit by a massive body by velocity v. At time t_1 compressive stress $\sigma_1 = E\ v/c$ acts in the whole bar. Due to this stress, the bar becomes shorter by $\Delta L = L\sigma_1/E = Lv_0/c$. This shortening is identical with the displacement of the massive body during this time, $u_1 = v\,t_1 = v_0\,L/c$. Similar relations also hold during further compression of the bar. The massive body at the left end moves with constant velocity, but the stress here increases in steps $2\sigma_0$ in the intervals $2\Delta t$, as shown in Fig. 3.2. The picture shows the “accurate” stair-like curve and the stress growth proportional to time according to the quasistatic theory. At impact duration longer than five-times the time for the stress wave travel through the body and back, the difference between both solutions is negligible.

3.2 Elastic waves in a bar hit by a body of finite size

The reality is more complex. The body, which hits the free end of the bar, has finite mass M and starts to slow down. The solution of this problem by Timoshenko [3] will be shown here. Denoting the mass of the body per unit of the cross section of the bar ($=M/S$) as m_1, the compression stress at the end of the bar as σ, and instantaneous velocity of the body as v, we obtain the equation of motion

$$m_1 \frac{dv}{dt} + \sigma = 0 \,, \tag{3.6}$$

Using the relationship between velocity and stress,

$$\sigma_0 = v_0 \sqrt{(E\rho)} \tag{3.7}$$

we can rewrite the Equation (3.6) as

$$\frac{m_1}{\sqrt{E\rho}} \frac{d\sigma}{dt} + \sigma = 0 \;. \tag{3.8}$$

This is a differential equation of the first order. After separation of variables σ and t, we obtain after integration and rearrangement

$$\sigma = \sigma_0 \exp\left(-\sqrt{\frac{E\rho}{m_1}}\, t \right) ; \tag{3.9}$$

here σ_0 is the stress magnitude in the first instant of the contact, given by Equation (3.4) for $v = v_0$. The stress along the stress wave is thus not constant, but decreases in exponential way. Similarly to the previous case this wave is reflected from the stiff wall as compressive one and returns back to the left end; here it is again reflects, and so on. A detailed solution is given, for example, in [3].

3.3 Elastic waves in a three-dimensional body

Other influences exist as well. If a local force impulse acts at certain place of a massive three-dimensional body, stress waves start propagating from this place in all directions. If the impulse went out from inside the body, the front of the wave has spherical shape at the beginning; but at large distances the wave can be considered as plane. The state of stress is more complex (triaxial). In the direction of propagation, formula similar to Eq. (3.3) holds for the velocity of longitudinal wave, but also the Poisson number μ (= $-\ \varepsilon_{\text{transverse}}/\varepsilon_{\text{longitudinal}}$) plays a role:

$$c_1 = \sqrt{\frac{E(1-\mu)}{(1+\mu)(1-2\mu)\rho}} \tag{3.10}$$

The velocity c_1 is higher than the velocity c in a thin bar. This is because the lateral displacements in a three dimensional body are suppressed, while the longitudinal deformation in a bar is accompanied by lateral contraction. The ratio c_1/c depends on Poisson´s number. For $\mu = 0.2$ is $c_1/c = 1.054$, for $\mu = 0.3$ is $c_1/c = 1.160$, and for $\mu = 0.45$ is $c_1/c = 1.948$. For μ approaching to 0.5 (incompressible material) the velocity of stress propagation would approach infinity.

In addition to longitudinal waves also transverse waves exist, caused by the particles of the body moving perpendicularly to the direction of wave propagation. Their velocity is

$$c_2 = \sqrt{\frac{G}{\rho}} \; ; \tag{3.11}$$

G is the modulus of rigidity. Also surface waves (Rayleigh) exist and other. More to this topic can be found in [1 – 3, 6, 7], for example.

3.4 Plastic waves

If the stress caused by dynamic load, attains or exceeds the material´s yield strength, also plastic wave starts propagating from this point, in addition to the elastic wave. If the material exhibits linear strain hardening, the velocity of longitudinal plastic wave (i.e. velocity of growth of plasticised region) will be

$$c_{pl} = \sqrt{\frac{E_{pl}}{\rho}} \; ; \tag{3.12}$$

E_{pl} is the strain hardening modulus of the material (see Fig. 5.2a and formula 5.2b in Chapter 5). E_{pl} for construction metals is usually by two or three orders lower than Young modulus, so that the velocity of propagation of plastic waves is at least by one order lower than the velocity of elastic waves. For materials with nonlinear strain hardening it is necessary to replace E_{pl} in Equation (3.12) by the tangent $\mathrm{d}\sigma/\mathrm{d}\varepsilon$ to the curve $\sigma - \varepsilon$ at the investigated point. If the tangent modulus decreases with increasing strain, the longitudinal wave will propagate with lower velocity. In materials that strain-harden with increasing strain, the plastic wave will propagate by higher velocity. More to this topic can be found in [1, 6, 8].

3.5 Hydraulic shock

Sudden stopping of flow of a liquid in a tube causes a sudden increase of pressure, denoted as hydraulic shock. A similar peak of pressure arises if the liquid is suddenly brought to movement, for example if a moving body hits a hydraulic shock absorber, described in Chapter 6.8. This is caused by the fact that in liquids and gasses any pressure impulse can propagate – similarly to solids – only by finite velocity, equal the sound velocity. This velocity in a liquid, c_h, depends on its compressibility as [9]:

$$c_h = \sqrt{\frac{K}{\rho}} \quad ; \qquad (3.13)$$

ρ is the density of the liquid and K is the modulus of its compressibility. (Numerically it equals the pressure necessary for the volume change by 100%.) The corresponding pressure peak is (compare with Equation 3.4)

$$p_h = v_0 \sqrt{\rho K} \quad . \qquad (3.14)$$

This pressure acts at the first instant, and gradually it develops to the values corresponding to the quasistatic solution. If the nominal pressure in operation is much higher than p_h, the pressure increase due to hydraulic shock makes only part of the total pressure increase and is not very significant.

The problems of impacts and shocks can be solved by various methods, simpler or more complex. At the beginning, it is reasonable to determine the duration of the impact according to the quasistatic theory and to compare it with the duration for the passage of the elastic wave through the body. Also it is possible to compare the maximum stress according to the quasistatic theory and determined, for example, via the law of energy conservation (given in Chapter 2.7) with the stress calculated for the first instant of the impact, Equation (3.4). If the values according to the quasistatic theory are significantly higher than those according to the wave theory, the wave effects do not need to be taken into consideration and the quasistatic solution will be acceptable. In the opposite case, the wave character of stress increase must be considered.

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4. Courses of stopping for various resistances

This chapter will show the relationships among velocity, path, deceleration and forces, and the energy dissipated or accumulated in principal kinds of stopping, such as braking with constant resistance force, impact on a spring with linear characteristics (without energy dissipation), on a spring with linear characteristics and damping by shear friction, and impact on a spring with linear characteristics and damping proportional to velocity or damping proportional to the square of velocity. More can be found in literature [1 – 6].

4.1 Braking with constant resistive force

Constant force during braking can be achieved in several ways. It is, for example, friction with constant coefficient of friction, controlled deformation of specially shaped element made from metal, or gradual destruction of a honeycomb structure. Constant force can also be achieved by a hydraulic shock absorber of special construction, which will be described in Chapter 6.8.

The situation at impact is depicted schematically in Fig. 4.1. A body of mass *m* hits at time $t = 0$ by velocity v_0 at a bumper with a friction damper. During compression of the bumper a reaction force F_t = const arises, whose magnitude can be set up by adjusting the damper. The motion equation of the body is

$$m\ddot{x} + F_t = 0 \; ; \tag{4.1}$$

$\ddot{x}$ is the body deceleration; two dots above *x* denote its second derivative by time,

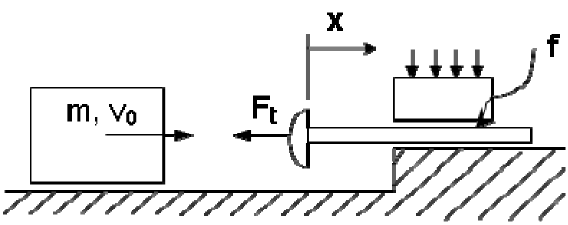

Figure 4.1. *Braking with constant force.* F_t *–force, f – coefficient of friction*

d^2x/dt^2. This equation can be rewritten to the form

$$\ddot{x} = -F_t/m \quad . \qquad (4.2)$$

Integrations of this expression give velocity ($\dot{x} = v$) and displacement (x) of the body:

$$\dot{x} = -(F_t/m)\,t + C_1 \quad , \qquad (4.3)$$

$$x = -(F_t/2m)\,t^2 + C_1 t + C_2 \;; \qquad (4.4)$$

C_1 and C_2 are integration constants, which can be determined from the initial conditions. In the investigated case these conditions are: $x(t{=}0) = 0$ a $v(t{=}0) = v_0$, so that it follows from Eqs. (4.3) and (4.4) that $C_1 = v_0$ and $C_2 = 0$, and thus

$$v(t) = v_0 - (F_t/m)\,t \qquad (4.5)$$

$$x(t) = v_0\,t - (F_t/2m)\,t^2 \;. \qquad (4.6)$$

Velocity during the braking decreases with time linearly from v_0 to 0 (dashed line in Fig. 4.2a). The stopping time according to Eq. (4.5) for $v(t_b) = 0$ is

$$t_b = v_0\, m/F_t \;. \qquad (4.7)$$

The path x grows with time slower according to quadratic parabola (solid curve in Fig. 4.2). The braking path can be obtained from Eq. (4.6). For the time $t = t_b$, expressed via Eq. (4.7) it is:

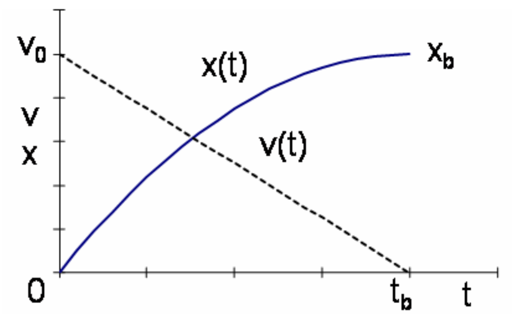

Figure 4.2. *Braking with constant resistance. Velocity v and path x as functions of time t.* v_0 *– initial velocity,* x_b, t_b *– braking distance and time*

$$x_b = ½\, v_0^2\, m/F_t \tag{4.8}$$

The deceleration during stopping is constant according to Eq. (4.2), and equal

$$a = -F_t/m\ . \tag{4.9}$$

The braking force, also constant, equals F_t. The velocity of the body decreases with the path x according to the expression

$$v = \sqrt{(v_0^2 - 2ax)}\ , \tag{4.10}$$

which is obtained if the acceleration is expressed as $\ddot{x} = d(\dot{x}^2)/(2dx)$ and integrated. The course is depicted in Fig. 4.3.

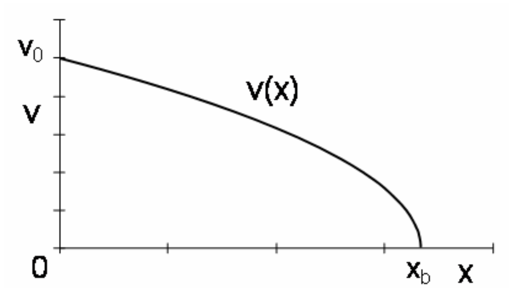

Figure 4.3. *Braking with constant resistance. Velocity v as function of path x.*

If a device for stopping of a moving body should be designed, Equation (4.9) can be used to determine the deceleration a for the allowable force F, or, vice versa, the necessary braking force F_t for the prescribed deceleration a. Equation (4.8) serves for the determination of the braking distance, or the braking force for the maximum allowable path. **The stopping with constant deceleration exhibits the lowest braking force and deceleration, and is, therefore, the most efficient!** This is obvious from Fig. 4.4, which compares the courses of braking forces at various cases of stopping. Except the stopping with constant deceleration (line “a”), any other case of braking exhibits higher values of braking force, either at the beginning or at the end of stopping. The efficiency of a stopping appliance can be evaluated according to the ratio of the force for constant deceleration and the maximum force needed with the investigated shock absorber,

$$F_{mean} / F_{max} = \eta_F \qquad (4.11)$$

In design of shock absorbers, η_F is called Crash Force Efficiency, CFE [2].

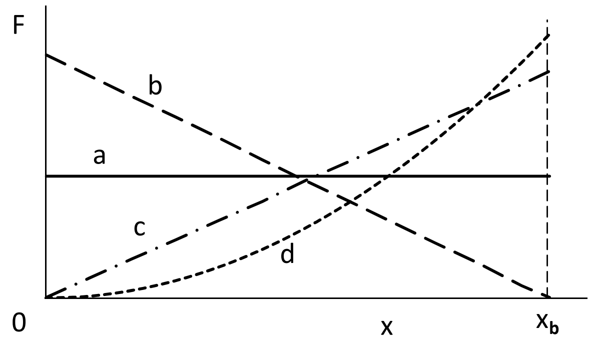

***Figure 4.4**. Courses of resistive forces in various cases of braking.*
a - constant force, b – force decreasing linearly with displacement,
c – spring with linear characteristics, d – force increasing with square of path

4.2 Impact on a spring with linear characteristics

The arrangement is depicted schematically in Fig. 4.5. At time $t = 0$ a body of mass m hits by velocity v_0 a spring of stiffness k, assumed constant. Compression of the spring generates a reaction force of magnitude

$$F = kx \; ; \qquad (4.12)$$

x is the displacement of the body, identical with the compression of the spring (curve „c“ in Fig. 4.4).

The motion equation of the body is

$$m\ddot{x} + kx = 0 \; ; \qquad (4.13)$$

$\ddot{x}$ is acceleration of the body. Equation (4.13) can be rewritten to the form

$$\ddot{x} + (k/m)\, x = 0 \; . \qquad (4.14)$$

This differential equation of the second order is the same as the well known equation of free vibrations of a mass point on a spring, which is usually written as

$$\ddot{x} + \omega^2 x = 0 \; ; \qquad (4.15)$$

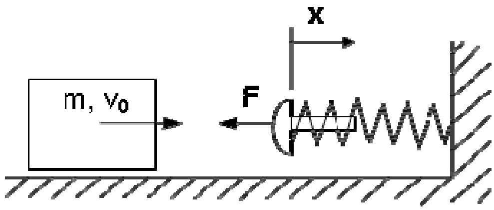

Figure 4.5. *Braking by a spring with linear characteristics*

ω corresponds to circular frequency (angular velocity), equal $\omega = \sqrt{(k/m)}$. General solution of Equation (4.15) is

$$x(t) = A\sin(\omega t) + B\cos(\omega t) = C\sin(\omega t + \varphi)\ . \tag{4.16}$$

A and B, or C and φ are constants, which can be determined from the initial conditions. For velocity $v(t)$ and acceleration $a(t)$, it holds

$$v(t) = \mathrm{d}x/\mathrm{d}t = \omega\ C\cos(\omega t + \varphi)\ =\ v_0\cos(\omega t + \varphi)\ , \tag{4.17}$$

$$a(t) = \mathrm{d}v/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2\ = -\omega^2\ C\sin(\omega t + \varphi) = v_0\omega\sin(\omega t + \varphi)\ . \tag{4.18}$$

In this case, at the beginning ($t = 0$) the displacement $x = 0$ and velocity $v(0) = v_0$. With these conditions, $\varphi = 0$ and $C = v_0/\omega$. The velocity decreases during the braking with time as a cosine function, and the displacement and deceleration increase as sinus functions (Fig. 4.6). As stopping of a moving body is investigated, only the first quarter period of the oscillating movement will be considered here. The velocity drops to zero for $\omega t = \pi/2$, that is in the time

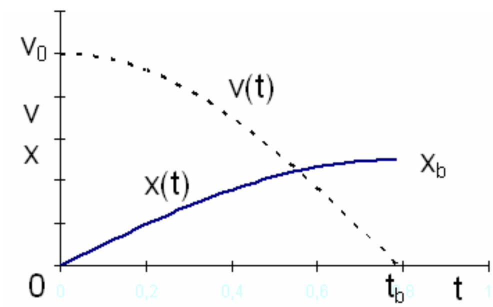

Figure 4.6. *Impact on a spring with linear characteristics*

$$t_b = \frac{\pi}{2\omega} = \frac{\pi}{2}\sqrt{\frac{m}{k}} \quad , \tag{4.19}$$

with the corresponding displacement

$$x_{max}(v = 0) = x_b = v_0/\omega = v_0\sqrt{(m/k)} \ . \tag{4.20}$$

As it follows from Eq. (4.12), the force during braking grows linearly with the path (line c in Fig. 4.4).

The maximum deceleration and maximum force will be attained at the end of braking ($t = t_b$, $x = x_b$), with the values

$$a_{max} = v_0\sqrt{(k/m)} \ , \tag{4.21}$$

$$F_{max} = m\, a_{max} = v_0\sqrt{(mk)} \ . \tag{4.22}$$

The maximum force and deceleration will be higher for higher initial velocity and stiffness of the spring k (more accurately, square root of it, $\sqrt{k}$; see also the example at the end of Chapter 2.

Let us look at the influence of the mass of the stopped body. It is of two kinds: the maximum deceleration decreases with the square root of m, i.e. $\sqrt{m}$, while the maximum force increases proportionally to $\sqrt{m}$. Four times heavier body means the deceleration drops to 50% and the maximum force at stopping is twice as high.

It is also useful to know the relationships among the energy to be dissipated and the individual quantities. The kinetic energy changes during stopping into the potential energy of the compressed spring, so that

$$E_{kin} = ½\, mv_0^2 = E_{pot} = ½\, F_{max}\, x_{max} = ½\, k\, x_{max}^2 = ½\, F_{max}^2 / k \tag{4.23}$$

These formulae enable one to determine the maximum force or braking path or demanded spring stiffness for the given energy of impact. However, one problem remains with this arrangement. The compressed spring has a tendency to rebound and throw the stopped body back by the initial velocity. A strong joining of the body and spring would cause their permanent oscillations with the frequency ω given above. If the body should stop at reaching the maximum compression, a suitable mechanism must be used here that will ensure it. It is also possible to utilise the energy dissipation during the braking, as described in Chapter 4.3.

Several springs in series

Until now we assumed that the body hits one spring, which will be compressed. However, often it comes to a collision of two bodies, and each is less or more compliant. An example is a collision of two cars or impact of a car on a compliant barrier. Sometimes a protective layer from a more compliant material was created on the surface of the body to be protected and this body itself has certain compliance even without this layer. Basically, we can imagine both parts as springs in series (Fig. 4.7a). How is this situation solved?

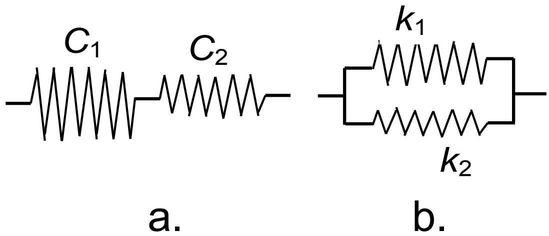

Figure 4.7. *Springs connected in series (a) and parallel (b)*

Generally, more springs can be arranged in series. In such case, the same force acts in each spring (generally, in each element in the series) and their deformations are summed up. Since the deformation is calculated as the product of force and compliance, the resultant compliance C of n springs in series equals the sum of compliances of the individual springs:

$$C = \sum_{j=1}^{n} C_j \,, \quad k = 1/C = \frac{1}{\sum 1/k_j} \quad ; \qquad (4.24)$$

C is compliance; its reciprocal, k, is stiffness; j-th spring is denoted by subscript j. For two springs in series:

$$C = C_1 + C_2 \,, \quad k = \frac{k_1 k_2}{k_1 + k_2} \quad . \qquad (4.25)$$

If none of the springs has limited deformation, the stiffness k of the system for the whole extent of deforming is described by Eq. (4.25). In some cases, however, one spring has limited stroke and the situation is more complex, as it will be shown further. The compressions of the individual springs are x_1, x_2, and the resultant compression is $x = x_1 + x_2$. The maximum possible compression of spring 1 is δ_1. As soon as spring 1 exceeds this deformation, with corresponding force

$$F_{\delta 1} = k_1 \delta_1 \tag{4.26}$$

and the total deformation

$$\delta = F_{\delta 1}/C\,, \tag{4.27}$$

the spring 1 does not act any more, and further increase of force is held by the spring 2 only. The total compliance of the system now corresponds only to this spring. Therefore, the stiffness has increased from the value k, given by Equation (4.25), to the value k_2. The situation is depicted in Fig. 4.8. The area below the curve Force – Displacement represents the work consumed or accumulated. In the first stage (for $x \leq \delta_1$) a simple relationship holds between the energy and force, and this makes possible easy determination of the displacement and force for the

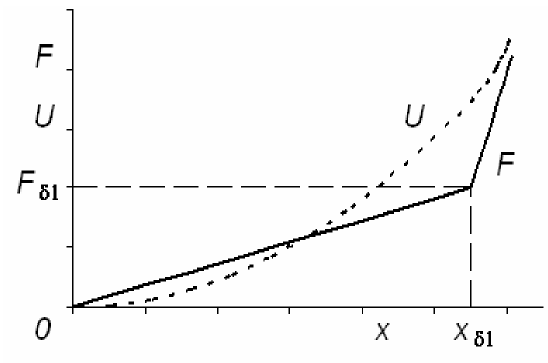

Figure 4.8. *Two springs in series. Force F and energy U as functions of compression x. $x_{\delta 1}$ – maximum possible compression of spring 1, $F_{\delta 1}$ – corresponding force*

known energy of the moving body. This was illustrated by an example in Chapter 2.7. If, on the other hand, the energy of the body to be stopped is higher than $\frac{1}{2}F\delta^2/k$, no explicit relationship between the energy and force exists. The force is

$$F = F_{\delta 1} + (x - x_{\delta 1})k_2 \tag{4.28}$$

and the consumed energy equals

$$E = ½\, F_{\delta1} x_{\delta1} + ½\, (x - x_{\delta1})(F + F_{\delta1})\ . \tag{4.29}$$

If the force or displacement should be determined for the energy to be absorbed, the easiest way is to calculate the energies for various values of the path x. Also Excel can be used for this purpose. The pertinent table can be used for the determination of the path corresponding to the dissipated energy.

REMARK. With **parallel springs** (Fig. 4.7b), the forces of individual springs are added together and the resultant stiffness equals the sum of their stiffnesses,

$$k = \Sigma k_j\ . \tag{4.30}$$

4.3 Impact on a spring with linear characteristics and friction damping

The situation is depicted in Fig. 4.9. The body of mass m and velocity v_0 hits a spring, which is joined with an additional friction damper, whose resistance F_t is constant. The motion equation for the decelerated body is

$$m\ddot{x} + kx + F_t = 0\ ; \tag{4.31}$$

k is the spring stiffness (N/m) and F_t is friction force (N), for example $F_t = N f$, where N is the force pressing the brake pad in a friction damper to the solid counterpart, and f is the coefficient of friction, assumed constant.

After a rearrangement, we obtain the following equation for the movement (again $\omega^2 = k/m$):

$$\ddot{x} + \omega^2 x = -F_t / m\ , \tag{4.32}$$

This is nonhomogeneous differential equation, i.e. with nonzero right side. The solution is obtained as a sum of the solution of the homogeneous equation and so-called particular integral,

$$x(t) = x_{hom} + x_{part}\ . \tag{4.33}$$

The solution of homogeneous equation (without the right side) is the same as in the previous section [Eq. (4.16)], and the particular integral is a function that satisfies

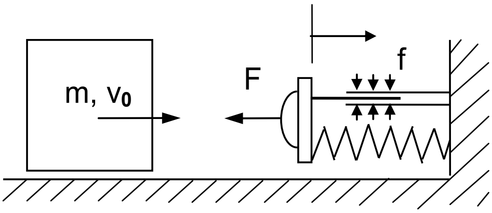

Figure 4.9. *Braking with a linear spring and friction damper.*
F – total force from the spring (= kx) and friction damper (= F_t)

the complete equation (4.32). The resultant solution has the form

$$x(t) = C\sin(\omega t + \varphi) - F_t/(m\omega^2) \;; \tag{4.34}$$

$\omega = \sqrt{(k/m)}$ is circular frequency and C and φ are constants, which will be obtained from the initial conditions. The solution for $t = 0$, $x = 0$ and $\dot{x} = v_0$ is

$$x(t) = \frac{v_0}{\omega}\sin(\omega t) - \frac{F_t}{m\omega^2} \;; \tag{4.35}$$

the angle φ is in radians. Insertion of both constants into Eq. (4.34) will yield the time course of the path during braking. The velocity decreases with time as

$$\dot{x}(t) = \omega C\cos(\omega t) \,. \tag{4.36}$$

Both courses are similar to those depicted in Figure 4.6.

The time for the drop of velocity to zero is obtained from the condition $\cos(\omega t_b) = 0$:

$$t_b = \frac{\pi}{2\omega} \tag{4.37}$$

The braking path corresponding to this time is

$$x_b(v = 0) = \frac{v_0}{\omega}\sin(\omega t_b) - F_t/(m\omega^2) \,. \tag{4.38}$$

This path is also equal to the maximal compression of the spring. The corresponding force $F_{max} = kx_b$ will try to return the body back, and this must be avoided by a suitable means.

Generally, the braking force is proportional to the compression of the spring; it thus increases proportionally to the path (curve "c" in Fig. 4.4).

A part of the initial kinetic energy $E_{kin} = ½\ m\ v_0^2$ was dissipated by friction and another part remained accumulated in the spring. The dissipated energy W_{dis} and accumulated energy of elastic stresses E_{el} are

$$W_{dis} = F_t\, x_b\,, \qquad E_{el} = ½\, F_{max}\, x_b = ½\, k\, x_b^2\ . \tag{4.39}$$

The total work consumed in the braking equals the sum of both components,

$$E_{tot} = W_{dis} + E_{el}\ . \tag{4.40}$$

This relationship can be used for the determination of the braking distance x_b and maximum braking force F_{max} for the given energy of impact E, spring stiffness k and the force of the damper, or for the finding of the necessary stiffness of the spring for the given braking distance or other parameters.

4.4 Impact on a spring with linear characteristics and damping proportional to velocity

The arrangement is depicted in Figure 4.10. It differs from the previous case by the damper whose resistance is directly proportional to the velocity of the body. Such case is typical for common dampers of vibration that use a liquid.

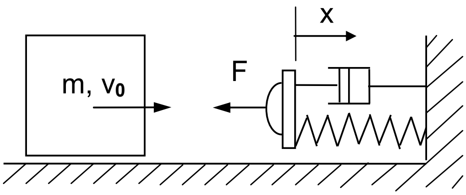

Figure 4.10. *Braking with a linear spring and damping proportional to velocity*

The motion equation for the decelerated body is

$$m\ddot{x} + b\dot{x} + kx = 0\ ; \tag{4.41}$$

k is the spring stiffness (N/m) and b is the damper constant (Ns/m), giving the resistive force of the damper at velocity 1 m/s. After a rearrangement we get

$$\ddot{x} + (b/m)\dot{x} + (k/m)x = 0\ , \quad \text{resp.}\ \ddot{x} + 2N\dot{x} + \omega^2 x = 0 \tag{4.42}$$

The solution can be sought in the form $x = Ce^{\lambda t}$, where λ is a constant. (Generally, two solutions exist, with constants C_1, λ_1, and C_2, λ_2.) With x expressed in Equation (4.42) in this way we get the following expression for the calculation of constants λ_1, λ_2 from the damper parameters:

$$\lambda_{1,2} = -N \pm \sqrt{N^2 - \omega^2}\ ; \tag{4.43}$$

here $N = b/(2m)$, $\omega^2 = k/m$. For vibrating movement, ω denotes natural circular frequency of free vibrations, and N denotes the "frequency" of damping. With respect to mutual relation of N and ω, three general cases can appear:

1. $N < \omega$ (subcritical damping)

The roots of Eq. (4.42) are complex conjugate, and the general solution is

$$x(t) = C\,e^{-Nt} \sin(\omega_1 t + \varphi_0)\ , \tag{4.44}$$

where C and φ_0 are constants, which will be found from initial conditions. For $x(0) = 0$ and $\dot{x}(0) = v(0) = v_0$, the solution is

$$x(t) = \frac{v_0}{\omega_1} e^{-Nt} \sin(\omega_1 t)\ . \tag{4.45}$$

The expression (4.45) has two components: exponentially decreasing term e^{-Nt} and sinus term [sin $(\omega_1 t)$], where ω_1 is the natural circular frequency of the vibrations with damping, which is related with the natural frequency ω without damping as

$$\omega_1 = \sqrt{(\omega^2 - N^2)} = \omega\sqrt{(1 - \delta^2)}\ ; \tag{4.46}$$

$\boldsymbol{\delta} = N/\omega$ is so-called **relative damping**. NOTE: Damped vibrations are somewhat slower than those without damping.

The sinus component in Eq. (4.45) shows that the displacement will be alternately positive and negative, so that the braked body will oscillate periodically there and back, though with damping. The frequency of vibrations f_1 is related to the circular frequency ω_1 as

$$\omega_1 = 2\pi f_1\ . \tag{4.47}$$

More pronounced vibrations are unsuitable in stopping, so that this case does not come into consideration. Hydraulic dampers with the resistance proportional to velocity are used especially for damping of vibrations. This case will be addressed in more detail in Chapter 7.

2. $N = \omega$ (critical damping)

General solution is

$$x(t) = \mathrm{e}^{-N\,t}\ (C_1 + C_2\,t)\ ; \tag{4.48}$$

C_1 and C_2 are constants. The time course of the movement of the stopped body for initial conditions $x(0) = 0$ and $v(0) = v_0$ is

$$x(t) = v_0\,t\,\mathrm{e}^{-N\,t} \tag{4.49}$$

and velocity

$$\dot{x}(t) = v_0\,\mathrm{e}^{-N\,t}\,(1 - Nt)\ . \tag{4.50}$$

Figure 4.11 shows both courses. We can see that the moving body is slowing down; at certain point its velocity drops to zero, then the body starts moving back and after some time it stops. This behaviour is understandable: the moving body passes a part of its kinetic energy on the spring as potential energy, and this energy then forces the body to move back. Even this case is not desirable for the stopping of a single impact. Again, an additional mechanism had to be used, which ensures the complete stop when the velocity drops to zero.

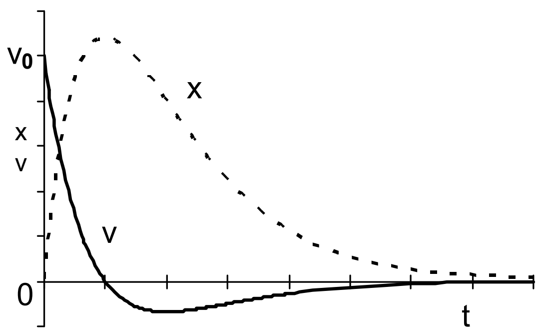

Figure 4.11. *Braking with critical damping (a schematic)*

Similar situation is in the following case with more pronounced damping.

3. $N > \omega$ (supercritical damping)

The time course of the path of the braked body for the initial conditions $x(0) = 0$ and $v(0) = v_0$ is

$$x(t) = (v_0/\alpha)\, e^{-N t} \sinh(\alpha t) \qquad (4.51)$$

and the velocity is

$$\dot{x}(t) = v_0\, e^{-N t} [\cosh(\alpha t) - (N/\alpha) \sinh(\alpha t)] \; ; \qquad (4.52)$$

α is defined by the expression

$$\alpha^2 = N^2 - \omega^2 \; . \qquad (4.53)$$

The time courses for critical and supercritical damping are similar to Fig. 4.11. The velocity decreases monotonously to zero and then it attains small negative values for some time. This means that the body at the end of its braking returns a small piece back. This, however, does not need to mean a serious problem, especially if similar mechanism is used for complete arrest as in the previous case.

4.5 Impact on a spring with linear characteristics and damping proportional to the velocity squared

The resistance proportional to the square of velocity is typical of dampers where the resistance is created by the flow of gas through and opening. The arrangement is the same as in Figure 4.10; only the characteristic of the damper is different. The motion equation is [1]:

$$\ddot{x} + \tfrac{1}{2} R\, \dot{x}^2 + \omega^2 x = 0 \; ; \qquad (4.54)$$

R [m^{-1}] is a constant characterising the damping, and $\omega = \sqrt{(k/m)}$ is circular frequency of free vibration; k is the spring stiffness and m is the mass of the stopped body.

The solution by elementary functions is not possible. However, if the acceleration is expressed by the derivative of the squared velocity with respect to the path,

$$\ddot{x} = d(v^2)/(2dx) = d(\dot{x}^2)/(2dx) \; , \qquad (4.55)$$

and the notation $\dot{x}^2 = z$ is introduced, Equation (4.54) can, after multiplication by two, be rewritten to the form

$$\mathrm{d}z/\mathrm{d}x \pm R\,z = -\,2\omega^2\,x\,. \tag{4.56}$$

The solution of this equation is

$$z(x) = C\,\exp(\pm\,Rx) \pm z_\mathrm{p}\quad , \tag{4.57}$$

where C is integration constant and z_p is particular integral. Its possible form is [1]

$$z_\mathrm{p} = [2\omega^2/R^2]\,(1 \pm Rx)\,. \tag{4.58}$$

The sign + holds for $\dot{x} > 0$, the sign minus is for $\dot{x} < 0$. For positive values $\dot{x}$ and the impact, where the displacement at the beginning ($t = 0$) is $x = 0$ and velocity $v(0) = v_0$, one obtains, after some algebra, the following expression for the velocity as a function of position x:

$$v = v_0\sqrt{[1 - (2w^2/(Rv_0^2))x]} = v_0\sqrt{(1 - Ax)}\ ; \tag{4.59}$$

A is the constant given by the expression $A = [2\omega^2/(Rv_0^2)]$. The path to arrest x_b is obtained from Equation (4.59) for $(v) = 0$:

$$x_\mathrm{b} = Rv_0^2/(2\omega^2)\,. \tag{4.60}$$

The time course of the motion can (with respect that $v = \mathrm{d}x/\mathrm{d}t$) be obtained by numerical integration of the modified expression (4.59).

Remark. From mathematical point of view, one solution of Equation (4.54) could express oscillatory movement. In our case, on the assumption that the stopping is achieved during the first half-period (using intensive damping with energy absorption), the movement back was not considered.

4.6 Changes of kinetic energy during braking

The task of a braking appliance in some cases is not to arrest the moving body fully, but to sufficiently reduce its kinetic energy, as the complete arrest will be achieved by other means. For such purpose it is useful to understand how the kinetic energy of the body decreases with decreasing velocity. Using the formulae for kinetic energy at the beginning of braking of the body with velocity v_0 and at the instant when the velocity dropped to v,

$$E_\mathrm{kin,0} = \tfrac{1}{2}\,mv_0^2\ ,\quad E_\mathrm{kin,v} = \tfrac{1}{2}\,mv^2 \tag{4.61}$$

one obtains

$$E_\mathrm{kin,v}/E_\mathrm{kin,0} = (v/v_0)^2 \tag{4.62}$$

For example, the drop of velocity *v* to the half of its initial value causes the decrease of kinetic energy of the body (and thus its possibility to cause damage) to one quarter; the drop of velocity to one tenth reduces the kinetic energy to one percent, and with the drop of velocity to three percent of the initial value the energy drops to less than one thousandth. In such case, even a harder dead stop can be acceptable.

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5. Response and damage of materials and components

5.1 Introduction

Analysis of load response needs the knowledge of relationship between stress and strain for the used material. On the other hand, this knowledge makes the choice of material for particular application easier. Figure 5.1 shows stress – strain diagrams ($\sigma - \varepsilon$) of tensile tests for various kinds of materials. The work expended for failure is proportional to the area below the stress – strain curve.

Figure 5.1a is typical of brittle materials such as glass, ceramics, quenched steel, or concrete. For low stress, direct proportionality exists between stress σ and strain ε,

$$\sigma = E\,\varepsilon\,; \qquad (5.1)$$

E is the modulus of elasticity in tension (Young modulus). Failure occurs if the maximum stress attains the ultimate strength σ_P. The failure is sudden, without

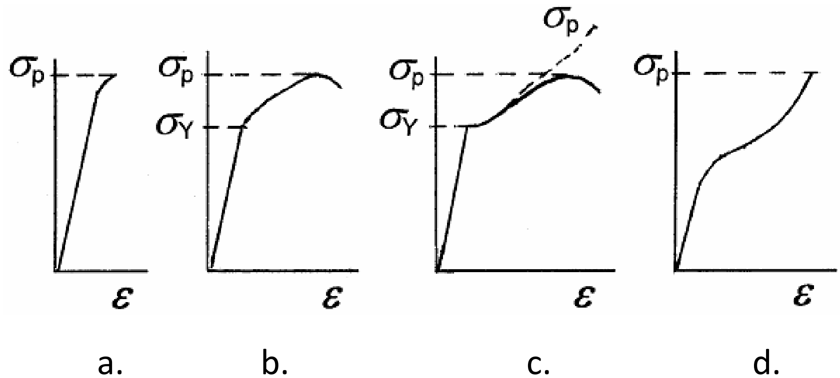


Figure 5.1. *Diagrams of tensile tests for various materials.*
a – brittle material, b, c – elastic-plastic materials, d – rubber.
σ – stress, ε – strain, σ_p – ultimate strength, σ_Y – yield strength

permanent deformations of the broken parts. The compressive strength is roughly one order higher than that in tension. Energy consumption till fracture is low.

Figures 5.1b, c are typical of ductile metal materials, such as soft structural steel, aluminium, copper and their alloys. For stresses lower than the yield stress σ_Y also here direct proportionality exists between stress and strain. As soon as the stress attains and exceeds the yield strength, the deformations start increasing faster, and permanent changes of shape and dimensions appear. Due to plastic deforming the energy consumption till failure is much higher than with brittle materials.

Figure 5.1d is typical for rubber and some other polymeric materials. The diagram is nonlinear for wider extent of loads, but no permanent deformations arise before fracture.

Now we shall look at the behaviour of the individual materials and components from them. We shall also look at the influence of shape and further factors.

5.2 Elastic – plastic response of ductile materials

The actual stress-strain diagrams are often approximated by simple expressions for easy understanding and calculations. Three most usual approximations are [1 – 3]:

Bilinear function (Fig. 5.2a):

$\sigma \le \sigma_Y \quad \varepsilon = \sigma / E \qquad \sigma_Y$ – yield strength (5.2a)

$\sigma > \sigma_Y \quad \varepsilon = \varepsilon_Y + (\sigma - \sigma_Y)/E'; \quad E'$ – strain hardening modulus (5.2b)

Power-law function (Fig. 5.2b),

$\sigma \le \sigma_Y \quad \varepsilon = \sigma / E$ (5.3a)

$\sigma > \sigma_Y \quad \varepsilon = K\sigma^m ; \qquad K, m$ – constants (5.3b)

Ideal elastic-plastic material without strain-hardening (Fig. 5.2c).

$\sigma \le \sigma_Y \quad \varepsilon = \sigma / E$ (5.4a)

$\varepsilon > \varepsilon_Y \quad \sigma = \sigma_Y \qquad \varepsilon_Y$ – strain at $\sigma = \sigma_Y$ (5.4b)

These formulae and diagrams are suitable for small strains. The stress at common tensile test is usually determined as the load divided by the nominal area of the cross-section, $\sigma = F / S$, and such diagrams are called conventional. The fact that –

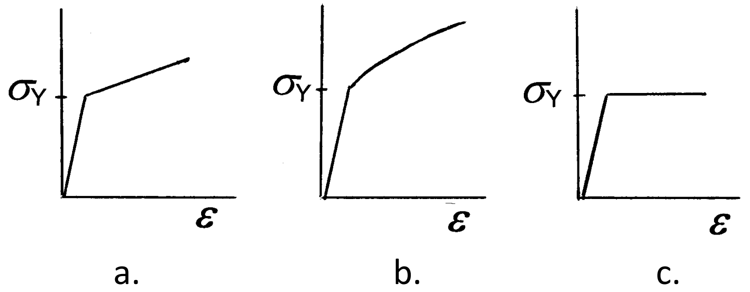


Figure 5.2. *Idealised diagrams. a – bilinear function, b – linear + power function, c – ideal elastic-plastic material without strain hardening;* σ_Y *– yield strength*

in addition to the elongation of the specimen – also its area changes, is usually neglected. This is allowable for deformations not exceeding several percent. The changes at larger deformations are not negligible, and the true stress differs from the nominal value. The relationship between the true stress and strain in tensile test of a soft structural steel is marked in Fig. 5.1c by dashed curve; the upper value of σ_P corresponds to the true stress at fracture. The differences grow especially at stresses approaching the ultimate strength, when a neck appears at the position of the future fracture (Fig. 5.3a). The situation in compression is opposite: the cross-section area becomes larger (Fig. 5.3b) and the true compressive stress is lower than the nominal one. Very ductile materials could sustain unlimited load in uniaxial compression. (Remember, for example, creation of foils from aluminium by rolling). Such behaviour must be respected in design of components for damping of impacts, where the permanent strains can be very large.

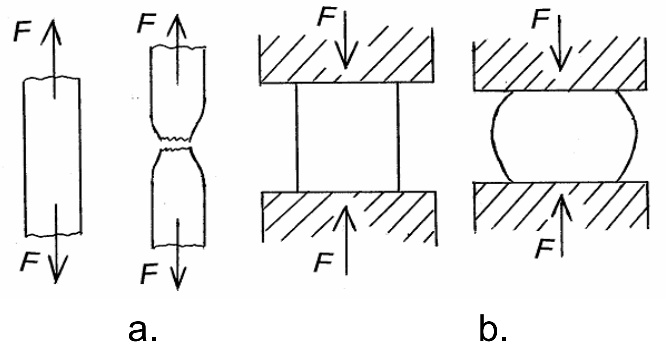


Figure 5.3. *Deformations of ductile materials under: a – tension, b - compression*

Further material characteristics of elastic-plastic materials

For elastic-plastic materials, used for components undergoing irreversible deforming in shock damping, also other characteristics are important in addition to strength: ductility, contraction and notch sensitivity [4], and fracture toughness for components with cracks. The last one will be treated later; here the first three are described.

Ductility is defined from the maximum strain in tensile test:

$$A = \frac{L_u - L_0}{L_0} = \varepsilon_0 \;; \tag{5.5}$$

L_0 is the initial length of the bar or its part, and L_u is the (ultimate) length of this part after fracture. The test bar is deformed not uniformly during testing; near the fracture section, called the neck, it elongates much more. Therefore, it must be considered how the ductility (5.5) was calculated. According to the standard [5] for material testing the ductility is given in percent and the subscript indicates whether the initial measured length equals five- or ten times the diameter of the bar (written as A_5, resp. A_{10}). If it is assumed that plastic properties of the designed component will play a role in the use, the material must have sufficient ductility. According to [5], A_5 = 15% or more is demanded for common steel structures, and the ultimate strength should be at least by 20% higher than the yield stress.

Contraction is defined as the largest relative reduction of the cross section area of the test bar, measured after the fracture,

$$Z = \frac{S_0 - S_u}{S_0} = \psi_0 \tag{5.6}$$

S_0 is the initial cross section area and S_u is the area of the smallest cross section of the broken bar. Z is given in percent.

Notch sensitivity is usually measured on a standard specimen with a notch, hit by a pendulum impact testing machine (Fig. 5.4). It is measured as the work K needed for the specimen breaking, divided by the area of the narrowest cross section, S_0,

$$KC = \frac{K}{S_0} \qquad (\mathrm{J/cm^2}) \tag{5.7}$$

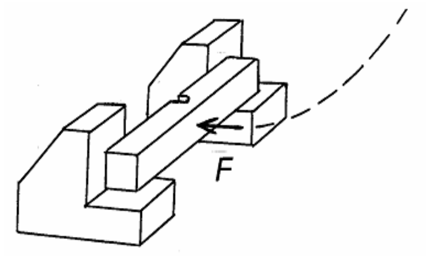


Figure 5.4. *Determination of notch sensitivity by pendulum hammer*

5.3 Load response of viscoelastic materials

Many components are made of polymeric materials. Here, the deformation depends not only at the load magnitude, but also on its duration and time course. Such materials are termed viscoelastic (VE), and many models exist for description of their behaviour [6]. These models are depicted as combination of *elastic elements* (with deformations directly proportional to the load or stress) and *viscous elements*, where stresses and forces are proportional to the velocity of deforming and the deformations grow with some delay. For elastic elements Hooke law is used, $\sigma = E\varepsilon$, while viscous elements are described by Newton law, $\sigma = \eta\dot{\varepsilon}$, where η is dynamic viscosity and $\dot{\varepsilon} = \mathrm{d}\varepsilon/\mathrm{d}t$ is strain rate. Two simplest models of VE materials are **Maxwell body** (M in Fig. 5.5) and **Kelvin-Voigt body** (K-V in Fig. 5.5). Maxwell body responds to load from the very beginning (the elastic part of deformation is instantaneous and is followed by the gradual growth of viscous part, which could grow in this model without limitation. The deformations of Kelvin-Voigt body grow from zero gradually and have finite magnitude. Often, more complex models are used, such as **standard linear solid** (= Kelvin-Voigt body in

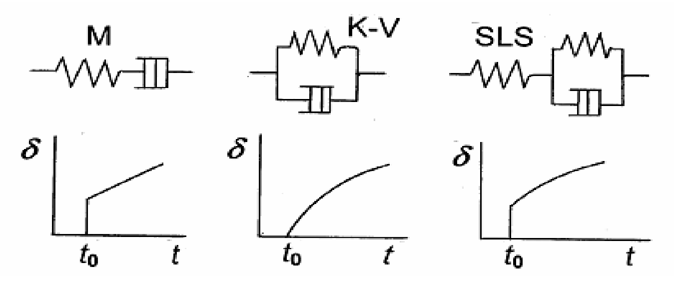


Figure 5.5. *Viscoelastic models. Deformation response to sudden load.*
M – Maxwell body, K-V – Kelvin-Voigt body, SLS – Standard Linear Solid

series with a spring, SLS in Fig. 5.5). This body can model the loads, which are constant or monotonically changing, but also vibrations with damping. With monotonic load the response function consisting of several terms is used. Often, it is described by means of a series of exponential functions, such as [6, 7]:

$$y(t) = F\left[C_0 + \sum_{j=1}^{n} C_j \exp\left(-t/\tau_j\right)\right] \; ; \tag{5.8}$$

F is the load, y is the deformation, t is the time, C_0, C_1,… are constants for given material and geometry of the body and load, and τ_j is the relaxation time, which characterises the rate of fading away of the load effects. Under fast load, the viscoelastic effects are usually small and for a short-time impact only the spring C_0 reacts. For steady-state vibrations the so-called phase angle φ is important, which characterises the shift between the load and deformation, as well as the energy losses; see also the hysteresis loop in Fig. 7.14 in Chapter 7.10.

5.4 Brittle and ductile failure

If the load attains the critical magnitude, failure of the body follows. Either as permanent change of its form (accompanied sometimes by fracture) with ductile materials, or fracture with materials brittle. A strict division of materials on brittle and ductile is not quite correct. Only a few materials can be termed really brittle, for example diamond or chalk. With many materials permanent change of shape can be achieved, though sometimes only in a very minute volume and extent, for example a permanent imprint made by a diamond in glass during microhardness testing [15]. It is more appropriate to speak about brittle or ductile fracture. Ductile fracture is accompanied by relatively large changes in shape. Brittle fracture does not exhibit signs of permanent change of shape. Many materials can fail by both modes, depending on the situation. Five **factors contributing** significantly to **brittle fracture**: 1. brittle material, 2. tensile stress, 3. low temperature, 4. impact or sudden load, and 5. complicated shape of the component, with notches or other stress concentrators. Some factors will be looked at here in more detail.

State of stress. Tensile stresses try to move the individual atoms apart, and this promotes the fracture by detachment, Fig. 5.6a. Shear stress moves the individual layers of atoms one along another, which promotes plastic deforming, Fig. 5.6b. However, one must not forget that under shear load also planes with tensile stress exist in the body, and tensile load generates also shear stresses in certain directions.

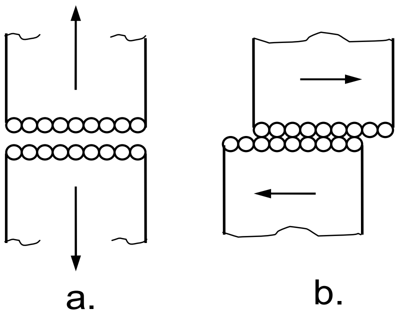


Figure 5.6. *Fracture by tearing-off (a) and permanent shape change by shear (b).*

Temperature. Permanent shape changes of metal bodies occur by movement of many atoms. These are more movable at higher temperatures and less movable at lower ones. For example, a part of soft steel, which can be deformed permanently by fast load even at room temperature, breaks if the temperature is lower than so-called **transition temperature**. This is the temperature where the notch sensitivity increases suddenly. Transition temperatures of various materials are different. They are very low for some metals, e.g. below −100°C, while for some other the transition occurs even at room temperature.

Stress concentrators. The stress in the vicinity of a sudden change of shape or cross section is high, but at some distance from it is much lower. Suitable conditions for plastic flow (and also for the absorption of impact energy) thus exist only in a small volume. The ability of plastic flow here is exhausted rather soon, in contrast to a component with constant or slowly changing cross section, where the plastic deforming can occur in much larger volume. This issue will be addressed later.

Loading rate. Movement of atoms (during plastic flow) into new equilibrium positions needs time. The stress in dynamic loading grows very quickly, so that better conditions can arise for tearing-off than for plastic deforming.

Dangerous are especially the loads occurring at mutual collisions of brittle bodies or impacts on them. They are dangerous because due to absence of plasticity nearly all energy of impact is changed into the energy of elastic stresses, so that these can attain very high values. With respect to the lower tensile strength and low fracture energy of brittle materials even a seemingly innocent impact can cause damage or even fracture of the body.

The behaviour at impact depends on many factors. In addition to dimensions and mass of both bodies also their physical and strength properties play a role, as well as the total compliance given by the material, shape and way of fixing; a part is also played by the character of mutual contact and initial velocity of impact.

As regards physical quantities, modulus of elasticity is important in particular. If one or both bodies are of ductile material, quantities characterising these properties are important, for example yield stress and the strength. Also the strength dependence on the duration of load action can be important. This phenomenon, known as static fatigue, is observed with glass, china and some kinds of ceramics. It is caused by very slow growth of the present microscopic defects under load and enhanced by corrosive action of environment, such as air humidity. In contrast to common strength tests, where the stress increases slowly, so that the time to fracture takes tens of seconds, the duration of load at impact is only 10^{-2} to 10^{-5} s, i.e. 1000 – 1000000 times shorter. Shorter time under load reduces the stress corrosion at the crack tip, which results in higher strength. The strength increase in these cases is not very high (the strength of glass under impact is about 1,2–2 times higher than that under slowly increasing load, and further shortening of the time has no more influence), but sometimes it can decide whether the object fails or not.

Deformations and failure of components from various materials

Regardless some uncertainty we shall use the terms ductile and brittle materials in their common sense.

5.5 Ductile materials – elastic-plastic bending

Components intended for energy absorption should not be loaded by tension. Bending is better. Here we shall show the evolution of stresses in this case.
Figure 5.7 shows a beam on two supports, loaded in the centre by a transverse force. The lower part shows the distribution of bending moments

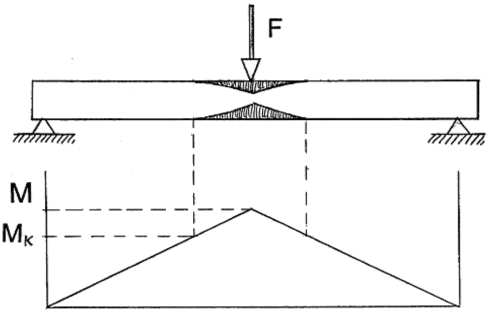


Fig. 5.7. *Elastic-plastic bending. Plastically deformed region and the distribution of moments. M - bending moment, M_K - moment, at which the yield stress is attained on the surface.*

Figure 5.8 shows gradual changes of stress distribution in the section in the middle of span. For simplicity we shall investigate the rectangular cross-section and ideal elastic-plastic material without strain-hardening (Fig. 5.2c) and with the same yield stress both in tension and compression. This is considerable simplification, but acceptable for giving general insight.

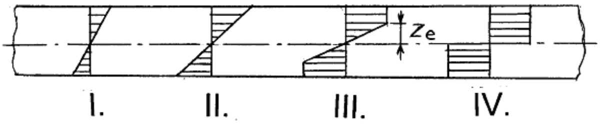


Figure 5.8. *Elastic-plastic bending.*
Stress distribution in the most strained cross-section during load increase (I , II, III, IV). z_e *– border between elastic and plastic deformations.*

At the beginning (Fig. 5.8-I) material deforms only elastically, and the stresses in the cross-section are proportional to the distance from the neutral axis. The stresses are tensile in one half of the cross-section and compressive in the other, equal [8]:

$$\sigma(z) = \frac{M}{J} z \quad . \tag{5.9}$$

M is the bending moment and J is the moment of inertia in bending (for rectangular cross-section, $J = bh^3/12$; h is the height of the cross-section and b is its width). Similar stress distribution exists till the maximal stress on the surface attains the yield strength σ_Y (Fig. 5.8-II). The corresponding moment and load are

$$M_K = \sigma_Y Z = \sigma_Y \frac{bh^2}{6} ; \qquad F_K = \frac{4 M_K}{l} , \tag{5.10}$$

Z is the section modulus in bending and l is the span of the beam. During further load increase the stress distribution changes. At places where the stress has attained the yield strength, the stress does not increase any more (material without strain hardening, Fig. 5.2c, is assumed). The moment increase above M_K is therefore transferred only by the material from the region with the stress still lower than σ_Y, and deformation will therefore increase faster. Near the neutral axis the elastic core exists with stress growing linearly with the distance from this axis (Fig. 5.8-III). In larger distances than z_e the material is fully plasticised, and the stress here is constant, equal the yield strength σ_Y. The transition between the elastic core and plastic region is at the distance z_e from neutral axis [1 – 3, 8]:

$$z_e = \sqrt{3\left(\frac{h^2}{4} - \frac{M_{el-pl}}{\sigma_Y b}\right)} \quad ; \tag{5.11}$$

M_{el-pl} is elastic-plastic moment, higher than M_K. Increase of the load causes further increase in thickness of the plasticised regions and reduction of thickness of elastic core. This core disappears when the bending moment attains the ultimate value

$$M_M = \sigma_Y \frac{bh^2}{4} \quad ; \tag{5.12}$$

Now the cross section is fully plasticised (Fig. 5.8-IV). The ultimate moment of rectangular cross section is by 50% higher than the moment corresponding to first attaining the yield stress. The deflection of the beam is still very small at this instant. The moment M_M could lead to unlimited rotation of both arms of a beam from a nonstrengthening material, and thus to the collapse of the structure. (With real materials some strain hardening occurs.) As the deformations are concentrated in the region of maximum moment, we speak about **plastic hinge** here (Fig. 5.7).

REMARK. As it is obvious from equations (5.10) and (5.12), plastic hinge arises relatively soon after the condition for plastic flow was exceeded. Also the deformations corresponding to the inception of plastic hinge are relatively small.

Similar situation exists with other shapes of the cross section. The ultimate bending moment can be expressed in the form

$$M_M = \sigma_Y Z_{pl} \; ; \tag{5.13}$$

Z_{pl} is so-called **plastic section-modulus**. The formulae for calculation can be found in Table 5.1 and literature, for example [9 – 11]. The proportion between the ultimate moment (in the plastic hinge) and the moment corresponding to the onset of plastic flow is the same as the ratio of the plastic and elastic modulus, $M_M/M_K = Z_{pl}/Z$. This ratio also characterises the reserve of load carrying capacity after the yield strength has been reached.

Table 5.1. *Elastic and plastic section moduli in bending for important shapes*

Section shape	*Z*	Z_{pl}	$Z_{pl} / Z = M_M / M_K$	
Rectangle	$bh^2/6$	$bh^2/4$	1.5	*b* - width, *h* - height
Circle	$\pi D^3/32$	$d^3/6$	1.7	*D* - diameter
I - profile			≈ 1.15	

If plastic flow appeared during the loading, residual permanent deformations remain in the body after unloading. Elastic component of deformations disappears. However, if the stress was distributed non-uniformly in the cross section, such as, for example, in bending, also residual stress σ_{res} remains in the body after unloading. The magnitude of this stress is obtained if fictive elastic stress $\sigma_{f\text{-}el}$ is subtracted from the true elastic-plastic stress $\sigma_{el\text{-}pl}$. The fictive stress $\sigma_{f\text{-}el}$ is such stress, which would act if no plastic deformations would have arisen, i.e. as if the body had very high yield strength. The residual stress can be calculated as:

$$\sigma_{res} = \sigma_{el\text{-}pl} - \sigma_{f\text{-}el} \tag{5.14}$$

Figure 5.9 shows the distribution of residual stress in the cross section after the elastic-plastic stress under load was distributed according to Fig. 5.8-III C. It is useful to know that the residual stresses in the place of the onset of plastic flow will have the opposite sign than the stresses under load.

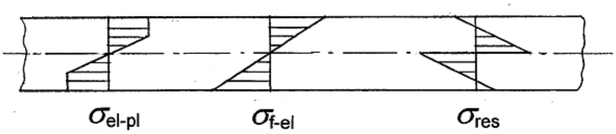


Figure 5.9. *Elastic-plastic bending. Determination of residual stresses σ_{res}. $\sigma_{el\text{-}pl}$ - stress distribution in elastic-plastic state, $\sigma_{f\text{-}el}$ - fictive elastic stresses*

During plastic flow much more energy is absorbed (dissipated) than during elastic deforming. The components designed for the reduction of impact effects by plastic flow, must, therefore, sustain large deformations without fracture; that is, they should have high ductility, which is here much more important than high strength. Bending is often used in metal parts for impact mitigation, so that they must sustain very large angles of bending without fracture.

5.6 Loss of stability by buckling

Specific kind of failure is the loss of stability of components by buckling. The situation can be illustrated on an example of a slender bar loaded by axial compressive force (Fig. 5.10a, b). Under lower load, the bar has straight axis, and the compressive stress in the cross section is distributed uniformly, with magnitude

$$\sigma_{tl} = F / S \ ; \qquad (5.15)$$

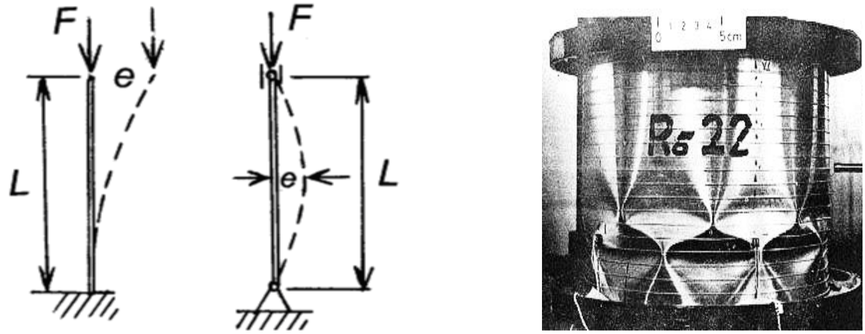
F is axial force and S is the cross-section area. If, however, the force reaches so-called **critical value** F_{cr}, the bar losses from any minute reason the stability and buckles (as shown by the dashed curve in Fig. 5.10a, b), and bending stress also appears here, which can be many times higher. This stress has the maximum value

$$\sigma_{max} = F\, e\, /Z \ ; \qquad (5.16)$$

e is the deflection. For example, the total maximum stress in the outer fibre of a bar with circular cross section is

$$\sigma_{max} = \sigma_{tl}\, [1 + 8(d/D)] \ ; \qquad (5.17)$$

D is the bar diameter, d is its deflection, and σ_{tl} is the compressive stress before buckling. For example, the same deflection as the bar diameter causes nine times higher stress than before buckling. Deflection quickly increases the value of e, so



a) b) c)

Figure 5.10. *Loss of stability by buckling. a, b) bending of bars and other slender parts, c) local loss of stability of an axially compressed cylindrical shell. [from mdp.eng.cam.ac.uk; wikipedia.com; 26.1.2018].*

that the bending moment continues increasing. This leads to fast attaining the yield stress, plastic flow and maximum possible deformation of a ductile material, and fracture of a brittle material. The whole process is usually instable and often ends by a collapse of the structure.

Critical load, at which the bar buckles in an ideal case, is [8, 9, 11,12]:

$$F_{cr} = \pi^2 EJ/l_0^2 \,, \qquad (5.18)$$

and the corresponding **critical stress** (F/S) is

$$\sigma_{cr} = \pi^2 E/\lambda^2 \,. \qquad (5.19)$$

Here E is the Young modulus, J is the inertia moment of the cross section in bending, l_0 is the characteristic length of the bar, which considers its true length and also the fixing of its ends, and λ is so-called slenderness ratio ($=l_0/i$, where i is the radius of gyration). The conditions for buckling of plates and shells can be expressed in similar way.

Formulae (5.18) and (5.19) hold for ideal cases: perfectly straight bar and compressive force acting in its axis, which passes through the centre of gravity of the cross section. These conditions are never totally fulfilled. Due to various imperfections the buckling starts at much lower load than F_{cr}. Nevertheless, the formulae are useful, as they say generally that

the resistance to buckling is lower for slender elements and low material stiffness.

The slenderness of an element is characterised by the ratio of its length and moment of inertia (or, simply, the thickness). The material stiffness is characterised by the modulus of elasticity.

Formulae (5.18) a (5.19) may also be used if one wants that the element will collapse at a load not exceeding certain demanded value.

Similar relationships with the modulus of elasticity, wall thickness, and characteristic length hold also for the loss of stability of plates loaded by compressive load in its central plane, or of shells. Here, also the local loss of stability can occur, for example during bending, but also during compression of an open profile or a thin wall tube loaded by axial compressive force (Fig. 5.10c). The geometric parameter is the ratio of the wall thickness and the radius of the tube. This phenomenon, local buckling in thin-walled tubes, is used in the construction of one-shot absorbers of energy at impact, for example in trains or cars (Fig. 6.4). More details can be found in Chapter 6 and in literature, for example [19 - 22].

5.7 Influence of notches and other stress concentrators

The shape of real parts is often complex, with sudden shape changes and notches (Fig. 5.11). It is important to know that the **bodies with notches** and other stress raisers are more prone to brittle fracture, i.e. with low energy consumption, even if they are made from a relatively ductile material. This is understandable from figure 5.12. The stress in the notch region is distributed non-uniformly. It is highest in the notch root and decreases with increasing distance from it. The detailed distribution of stress can be obtained by computer analysis. An approximate value of the maximum stress on the surface is calculated by a simple formula

$$\sigma_{max} = \alpha\,\sigma_{nom} \quad ; \qquad (5.20)$$

σ_{nom} is the nominal stress in the notch region, and α is the **stress concentration factor**. The values of these form factors for technically important shapes of notches and kinds of loading can be found in various handbooks, for example [12, 13].

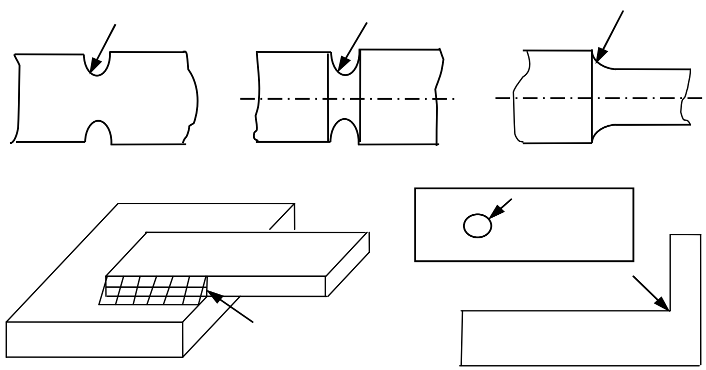


Fig. 5.11. *Examples of notches in components. The arrows show dangerous places.*

A component of brittle material fails if the maximum stress at certain place attains the material´s strength σ_P. It follows from Equation (5.20) that a brittle component with a notch fails if the nominal stress reaches the value

$$\sigma_{nom} = \sigma_P / \alpha \, . \qquad (5.21)$$

This means that a notch reduces the technical strength α–times !

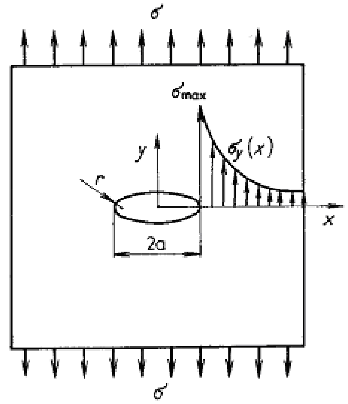


Figure 5.12. *Plate with and opening and stress distribution at one end (a schematic).*

Ductile materials deform elastically at the beginning. If the stress at some place exceeds the yield strength, the material here deforms plastically and more energy is absorbed here. The plastic flow in components, containing a notch with nonhomogeneous stress distribution, is limited only to small volume of material around the notch. The total energy absorption during an impact is therefore smaller than it would be in a body without a notch, where homogeneous stress distribution creates better conditions for plastic flow in much larger volume. (Compare, in Fig. 5.2, the area corresponding to elastic deforming (i.e. with the stress lower than σ_Y) with the area of the whole diagram including plastic deformations.) Also, the energy of impact is distributed easily over large volume in a body of simple shape, so that the corresponding stresses will be relatively low. In a body with a notch, on the contrary, the same amount of energy is concentrated in much smaller volume, so that here much more intensive plastic flow occurs, up to the exhaustion of deforming ability of the material and to fracture. The influence of a notch can be illustrated via impact on a sample without a notch and with a notch as common in testing the notch sensitivity (Fig. 5.4). If the sample is smooth, the falling hammer bends it, while a sample with a notch is broken with signs of brittle fracture.

Very dangerous are **cracks**, especially if tensile stress acts in the component. The failure of bodies containing cracks will be discussed in more detail in the next section. If one wants to avoid brittle fracture of a component from ductile material, all cracks and sudden changes of shape and stress concentrators should be avoided.

5.8 Influence of cracks, principles of fracture mechanics

Resistance of a body to impact is significantly reduced by cracks. One or more of cracks could have arosen during manufacture or during operation, for example due to fatigue under repeated loading. A body with a crack breaks easily in brittle manner at impact. A crack represents a very strong stress concentrator that significantly reduces plastic deforming (and thus also energy absorption) in the pertinent region. The first theoretical analyses, made under the assumption of purely elastic response, gave infinitely high stress in the crack root, which is impossible. Later, two approaches were proposed, which have overcome this drawback [14, 15]. The first one works with concept of **stress intensity factor *K***. This factor characterises the influence of nominal stress σ_{nom} in the crack region by the following relationship:

$$K_I = \sigma_{nom}\, Y \sqrt{l} \quad ; \qquad (5.22)$$

l is the length or any other characteristic dimension of the crack, and Y is a factor characterising the influence of the shape and position of the crack, its relative size with respect to the cross section area there, and also the character of stress distribution (tension, bending, etc.). The dimension is $Pa.m^{1/2}$ or $MPa.m^{1/2}$. The subscript I, II or III at K denotes the mode of crack opening (Fig. 5.13). The crack starts growing quickly if the stress intensity factor attains critical value; that is at

$$K \geq K_C \; . \qquad (5.23)$$

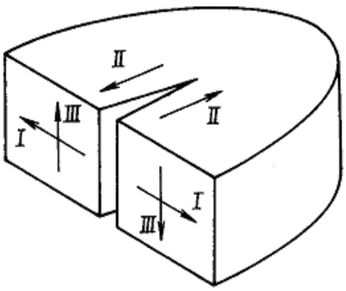


Figure 5.13. *Basic modes of crack opening.*

Most important is the simple crack opening (mode I). The corresponding critical value, K_{IC}, is called **fracture toughness**. This is measured on standard specimens containing a crack, with well-known relationship between the crack length and values of K_I. In the test, the specimen is loaded by increasing load, and the instant is found when the crack starts propagating quickly. The pertinent crack length and load are used for the calculation of K_I, which now corresponds to the fracture toughness K_{IC}. A possibility of fast fracture of another body with a crack is checked by comparing the value of stress intensity factor K_I for this body (with particular crack and load) with fracture toughness K_{IC}, measured (elsewhere) for this material.

REMARK. The formulae or diagrams for the determination of stress intensity factor can be found in various handbooks [16] or determined using a suitable computer model.

A component with a crack fails if the stress intensity factor at certain place attains critical value K_{IC}. From equations (5.22) and (5.23) it follows that the failure occurs if the nominal stress reaches the critical value

$$\sigma_{cr} \geq \frac{K_C}{Y\sqrt{l_{kr}}} \quad . \qquad (5.24)$$

As the shape factor Y depends also on the crack size, the critical length in some cases must be found by an iterative procedure. Vice versa, it is possible to find the critical crack length l_{cr} for the assumed nominal stress σ_{nom} as follows:

$$l_{cr} = \left(\frac{K_C}{Y\sigma_{nom}} \right)^2 , \qquad (5.25)$$

and to verify or ensure that the cracks in the component are smaller than l_{cr}.

The second approach to the assessment of the behaviour of an elastic body with a crack uses energy principles. If loaded, this body contains the accumulated energy of elastic stresses. If the crack grows, this energy is released gradually. On the other hand, the crack growth needs energy especially for plastic deforming of the material in the region of very high stresses in front of the crack tip and for creation of new fracture surfaces. The situation is depicted in Figure 5.14. The energy, consumed for the creation of new fracture surfaces grows directly proportionally with the crack length. The released energy grows with the square of this length. At the beginning, therefore, the energy needed for the crack growth prevails, but since a certain instant the energy released by this process will prevail, and the fracture process becomes unstable. For the description of fracture process, so-called **energy release rate *G*** was defined as the energy released by enlargement of the crack by

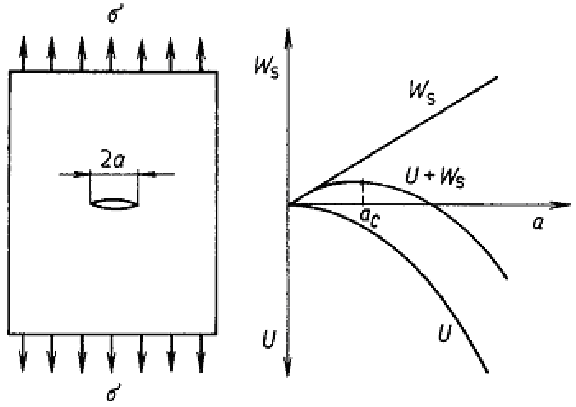


Figure 5.14. *A plate with a crack: energy balance[15].*
U – energy of elastic stresses, released by crack growth,
W_s – energy consumed by this growth, a_c – critical length

unit area (J/m^2), and **specific fracture energy Γ**, expressing how much energy is needed for the creation of fracture area of unit size (J/m^2). The condition of fast crack propagation is

$$G \geq \Gamma, \quad \text{resp. } G \geq G_C \; ; \qquad (5.26)$$

G_C means critical value of energy release rate.

Both approaches are equivalent in the evaluation of crack propagation in a brittle elastic body. For example, energy release rate for simple crack opening (mode I) is

$$G_I = K_I^2 / [E/(1-\mu^2)] \; . \qquad (5.27)$$

The formulae for other modes of crack opening are similar.

Table 5.2 on the following page gives values of fracture toughness and specific fracture energy for some materials.

The assessment of a possible failure uses commonly the stress intensity factor. However, the failure process should always be looked upon from energy point of view. For example, during an impact a certain amount of energy is passed to the body. And only if this amount is higher than the value $\Gamma \times S$, where S is the remaining area of the cross section of the body at the crack, complete fracture can be expected. Similarly, it can be predicted how much an existing crack becomes larger due to the impact, and what degradation of strength of the body it causes.

Table 5.2. *Fracture toughness K_{IC} and specific fracture energy G_C of some materials [17].*

Material	K_{IC} (MPa m$^{1/2}$)	G_C (J/m^2)
Steel	30 – 140	1000 – 85000
Grey cast iron	10 – 25	860 – 5400
Ceramics (various kinds)	1 – 20	2 – 2000
Glass	0.6 – 1,0	6 – 10
Epoxy resin	0.5 – 2,0	50 – 200

Now we shall look at failure of brittle bodies without visible cracks. We shall also see that failure can sometimes occur in various modes depending on the load conditions.

5.9 Failure of bodies from brittle materials

Failure of a component from ceramics, glass or another brittle material due to an impact by another body will depend on its velocity, on the geometry and dimensions of both bodies, and on their fixing or support. Sometimes it can therefore occur in various ways. The characteristic features of failure in two cases will be shown on an example of a beam-like body supported at both ends and hit by a ball of a strong material (Fig. 5.15).

As soon as both bodies come into contact, they start deforming at the region of contact. Moreover, the beam deflects. The velocity of the ball decreases and its kinetic energy changes gradually into the kinetic energy of the beam and the strain energy accumulated in both bodies. (Besides them, a part of energy is accumulated and dissipated at the beam supports.) Now, three cases can occur. If the energy of the flying ball is small, it all is converted into the strain energy, and the ball and the beam stop at certain deformation. However, they rebound immediately, the ball jumps back and the beam starts vibrating. (If the ball has fallen from certain height on a horizontal beam, the process will be repeated.) These vibrations will end after a while due to internal friction in the material and other energy losses. If the energy of impact is very high, the beam breaks before the ball stops. The ball and the parts of the beam can continue moving. In the third, immediate case, cracks arise, but the beam retains its integrity.

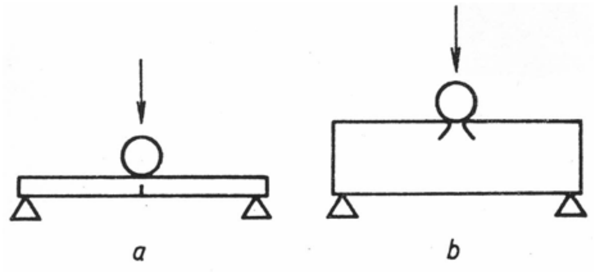


Figure 5.15. *Various modes of failure by impact [15]: a) due to bending stresses, b) due to contact stresses*

The damage of the beam can occur in two ways, depending on the dimensions of the beam and ball and on the conditions of impact [15]. In thin beams or plates bending stresses will prevail, so that the failure will be caused by bending (Fig. 5.15a). In thick beams with high bending stiffness the bending stresses will be low, and damage can be expected rather at the contact (for example, formation of a conical crack, Fig. 5.15b). In some cases, both kinds of damage occur.

If the velocity of impact is not high, the stresses can be determined from the formulae for stress, deformations and strain energy under static load, and with the assumption that the energy of impact will be changed into the energy of elastic stresses. In our case, bending and contact stresses are generated in the beam.

Bending stresses

If a concentrated force F acts in the middle of a long thin beam supported on its edges, it causes its deflection [15]

$$y = FC_o = F\, l^3/(48\, EJ_o)\ ; \qquad (5.28)$$

C_o is the bending compliance of the beam, l is its length, and J_o is the moment of inertia of the cross section in bending. The maximal bending (tensile) stress acts in the middle of the span, on the surface opposite to the force, and has the value

$$\sigma_o = F\, l\,/(4Z)\ ; \qquad (5.29)$$

Z is the section modulus of the beam in bending. The accumulated energy of bending stresses is

$$E_{pot,bend} = ½\, F\, y = ½\, C_o\, F^2 = l^3/(96\, EJ_o)\, F^2 \,. \qquad (5.30)$$

<u>Contact stresses</u>

If a ball of radius R is pressed into the plane surface of a massive body, a circular contact area of radius a is created here. The centre of the ball moves by [8, 15]

$$y = C_{cont}\, F^{2/3} = \left[9F^2/(16R_eE_e^{\,2})\right]^{1/3} \;; \qquad (5.31)$$

C_{cont} is so-called contact compliance, R_e is the equivalent radius of curvature of the contacting surfaces, and E_e is the equivalent modulus of elasticity of both materials in contact. Both constants can be obtained from the relationships

$$1/R_e = 1/R_1 + 1/R_2 \,, \qquad 1/E_e = (1 - \mu_1^{\,2})/E_1 + (1 - \mu_2^{\,2})/E_2 \qquad (5.32)$$

R_1 and R_2 are the curvature radii of body 1 or 2 at contact point, E_1 and E_2 are their moduli of elasticity, and μ_1, μ_2 are Poisson numbers. If one surface (of body 2, for example) is plane, $1/R_2 = 0$. If it is concave, R_2 has negative value. As it follows from Eq. (5.31), the deformation is not directly proportional to the force, but increases with its general root (= 2/3). The maximum tensile stress acts at the edge of contact surface and has the magnitude

$$\sigma_{cont} = [(1-2\mu)/3]\, p_0 \;; \qquad (5.33)$$

p_0 is the maximum pressure in the centre of contact area:

$$p_0 = 3/2\, [F/(\pi a^2)] = \pi^{-1}(6FE_e^{\,2}/R_e^{\,2})^{1/3} \;. \qquad (5.34)$$

Strain energy, accumulated in the contact region, is

$$E_{pot,\,cont} = 2/5\; C_{cont}\, F^{5/3} \qquad (5.35)$$

The total accumulated energy equals the sum of the energies of bending and contact stresses,

$$E_{pot} = E_{pot,\,bend} + E_{pot,\,cont} = 1/2\; C_oF^2 + 2/5\; C_{cont}F^{5/3} \qquad (5.36)$$

The energy of impact is

$$E_{imp} = ½\, m v_0^{\,2} \qquad (\text{or } E_{imp} = mgh\,)\,, \qquad (5.37)$$

where m is the mass of the ball, v_0 is its velocity at the instant of touching the beam, g is acceleration of gravity, and h is the height of the fall.

Under the assumption that the total impact energy is changed to the strain energy,

$$E_{imp} = E_{pot} , \tag{5.38}$$

one can (under the known compliances C_o and C_{cont}) determine from Eq. (5.36) the maximum force F and the corresponding bending and contact stresses, and check whether the component will sustain the impact, or not. Vice versa, it is also possible to determine the energy of impact, which a component of a known strength can survive. If the actual energy of impact is higher, the component breaks and the excess of energy is consumed for the creation of new fracture surfaces, acceleration of the broken parts, and further movement of the ball.

The danger of impact load will be illustrated on two cases according to [15].

Example 1. A steel ball 20 mm in diameter falls from height of 5 cm on a glass specimen having the shape of a beam 10 mm wide, 3 mm thick and 100 mm long. Determine the maximum stress at impact and compare it with the stress caused by the dead weight of the ball.

The material constants of the beam (1) and ball (2) are: ρ_1 = 2500 kg m^{-3}, ρ_2 = 7820 kg m^{-3}, E_1 = 70 GPa, E_2 = 210 GPa, μ_1 = 0.25, μ_2 = 0.3. The characteristic beam constants are $C_o = 13.2\times10^{-6}$ m.N^{-1}, $C_{cont} = 0.261\times10^{-6}$ m.N$^{-2/3}$, section modulus $Z = 1{,}5\times10^{-8}$ m^3. The ball mas is m_2 = 32.8 g.

In the static case, the beam load corresponds to the weight of the ball, i.e. $F_{stat} = m_2 g$ = 0.322 N. In the dynamic case, we shall use the energy of impact, $W = mgh$ = 16.1×10^{-3} J. The substitution of this value for U in Eq. (5.36) gives that the maximum force at impact will be F_{dyn} = 49.2 N, which is about 150-times higher than that caused by the dead weight ! The stresses will increase similarly: the maximum bending stress on static loading will be $\sigma_{o,stat}$ = 0.54 MPa; while it will be $\sigma_{o,dyn}$ = 82.0 MPa at impact. Maximum tensile stress at the periphery of the contact circle will be $\sigma_{cont,stat}$ = 20.9 MPa under the static load, and $\sigma_{cont,dyn}$ = 111.8 MPa at impact. For the sake of completeness it should be mentioned that the ball velocity was v_0 = 1.0 m.s^{-1}, and the impact energy was divided between the energies of bending and contact stresses as follows: $U_{o,dyn}$ = 16.0 mJ, $U_{cont,dyn}$ = 0.1 mJ.

REMARK. In these calculations several quantities were neglected: the own weight of the beam, the energy released from the ball during its movement in the gravitational field due to the beam deflection, and the energy accumulated at the beam supports.

The results indicate that whereas only negligible stresses arise under static load, dynamic load due to the same body causes stresses adequate for fracturing the glass specimen, where both its breakage as well as damage at the point of contact can be expected. We should also note that most of the impact energy was converted into bending stresses energy (but the contact stresses were higher than the bending ones). This holds quite generally for thin beams and other compliant bodies. The energy of contact stresses is therefore sometimes neglected in the calculation of dynamic forces, and simpler equation (5.30) is used instead of Eq. (5.36). On the other hand, the contact stresses alone can be considered when relatively rigid body is impacted.

Example 2. How the situation will change if the glass specimen from the previous example is hit by a sharp corundum particle with velocity v_0 = 1 m/s. The particle size is d = 0.1 mm and its tip radius is R = 0.01 mm? (The constants for corundum are ρ_2 = 4000 kg m^{-3}, E_2 = 370 GPa, μ_2 = 0.25, those for glass are E_1 = 70 GPa, μ_1 = 0.25. The corresponding constants of the contact are R_e = 1x10^{-5} m, E_e = 67.7849 GPa, a C_{cont} = 2.425×10^{-6} m/N$^{2/3}$.)

With regard to the minute weight of the particle ($m = \pi d^3 r/6 = 2.094\times10^{-9}$ kg) and energy ($E_{kin} = ½\, m v_0^2 = 1.047 \times 10^{-9}$ J) we can expect that the impact will have an effect in the contact region only. Equation (3.35) will give, after a rearrangement, that the maximum force at impact will equal

$$F = [5U_{kon}/(2\ C_{cont})]^{3/5} = [5\times1.047\times10^{-9}/(2\times2.425\times10^{-6})]^{3/5} = 0.016592\ \text{N}.$$

It is obvious that the bending stresses caused by so small force will be negligible.

The pressure in the centre of contact area, Equation (5.34) will be p_0 = 5015 MPa, and the tensile stress at its fringe, Eq. (5.33), will be σ_{cont} = 836 MPa. One can see that even a minute flying particle with sharp edges can bring about local surface damage at high impact velocity.

Major deviations from quasistatic theory will arise at higher impact velocities. The inertia of the body being struck will be involved on an increasing scale, together with the limited velocity at which it can deform. Both the deformation and the character of failure will change accordingly. Figure 5.16 shows three cases of damage of a glass plate whose centre was hit by a steel ball with various velocities. Figure 5.16a corresponds to relatively slow loading (up to impact velocity of about 1 to 2 m/s). The deflection curve at the instant when the fracture stress was attained is the same as in the static case (and also the character of the cracks formed and the energy consumption). Figure 5.16b corresponds to medium impact velocities (m/s up to tens of m/s). Due to inertial forces, which prevent the plate from deflecting rapidly, its deflection curve after the first contact will have a somewhat different shape compared to slow deforming, showing a larger curvature at the point of impact. The bending stresses are rather higher, so that the fracture will occur before the force F would attain the value causing fracture under static load. Also the energy accumulated at the instant of fracture will be smaller. (On the other hand, very short times under load can cause somewhat higher strength.)

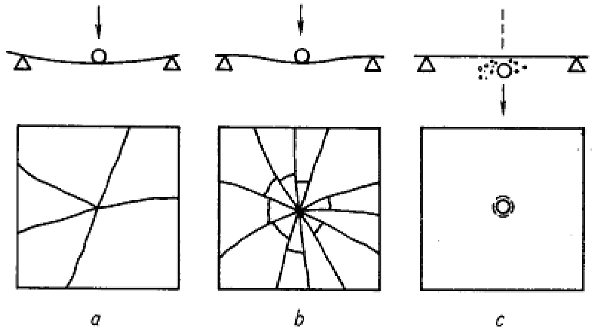


Figure 5.16. *Fracture of glass panes at various impact velocities.*

Figure 5.16c corresponds to high velocities of impact (of the order of hundreds m/s), for example due to a shot. Even before the plate starts to deflect, the contact stresses will attain values leading to the formation of a conical fracture passing throughout the entire plate thickness and a small piece in form of a truncated cone is knocked out from the plate [15]. In such a case the energy of bending is negligible and only the energy of contact stresses applies.

Many more cases of impact can arise in practice. The general rules involved will be similar to the examples given above.

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6. Structural elements for impact damping

This chapter shows how impacts are damped and energy absorbed using various kinds of elements, for example rings and systems of them, shells, honeycombs and other elements with cellular structure, inverted metal tubes, airbags and hydraulic shock dampers.

6.1 Rings and systems of rings

Efficient elementary parts for one-time absorption of mechanical energy are metal circular rings, which are deformed by point forces of radial direction (Fig. 6.1). These forces generate bending moments and stresses. The situation in the simplest case with two forces acting at the opposite points *A* on the ring of radius *R* is depicted in the figure. At the beginning only elastic deformations arise. Bending moment in the section containing angle φ with the direction of the force *F* is [1, 2]:

$$M(\varphi) = \tfrac{1}{2} FR\,(2/\pi - \sin\varphi) \quad . \tag{6.1}$$

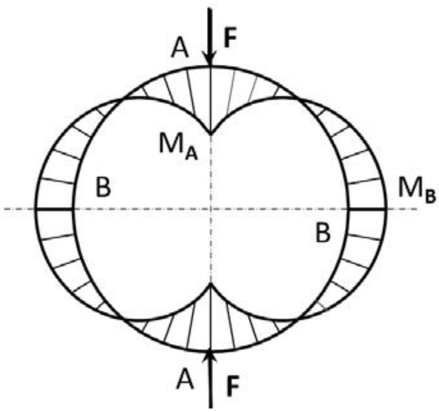


Figure 6.1. *Circular ring loaded by two forces at points A. Distribution of bending moments.*

The maximal bending moment and stresses act at the points of force application (section A; $\varphi = 0$):

$$M_A = (1/\pi)\,FR = 0.3183\,FR \tag{6.2}$$

The moment changes continuously to the point B, where it attains a lower maximum value of the opposite sense,

$$M_B = -0.1817\ FR\ ; \qquad (6.3)$$

see also Equation (6.10b) for $n = 2$. As soon as the bending moment in section A attains the value

$$M_Y = \sigma_Y Z\ , \qquad (6.4)$$

where Z is the elastic section modulus for bending of the ring cross-section and σ_Y is the material´s yield strength, plastic deformations appear in the outer fibres; in all other places the material is deformed elastically. For rectangular cross section, $Z = bh^2/6$, with b denoting its width and h the thickness in the force direction; for circular cross section, $Z = \pi d^3/32$; d is the diameter of the ring cross section. The corresponding load is

$$F_Y = (1/0.3183)\ M_Y / R = \pi\ \sigma_Y\ Z / R \qquad (6.5)$$

For simplicity, we shall assume here elastic-plasic material without strain hardening. With this material, the stress keeps the value equal the yield strength even for larger strains (Fig. 5.2c). The thickness of plastically deformed outer layers gradually increases (as explained in Chapter 5.5), and at the load

$$F_{pl,1} = \pi\ \sigma_Y\ Z_{pl} / R \qquad (6.6)$$

the elastic core disappears and the cross-section at A is fully plasticised; plastic hinge was formed here (see Chapter 5.5). The corresponding bending moment is

$$M_{pl} = \sigma_Y\ Z_{pl}\ ; \qquad (6.7)$$

Z_{pl} is so-called plastic section-modulus; for rectangular cross section, $Z_{pl} = bh^2/4$, and for circular cross section, $Z_{pl} = d^3/6$.

When plastic hinge at section A has been created, a larger part of the ring is still deformed only elastically, and the total deformations are small. During further load increase the stresses at plastic hinges remain constant and grow only out of them. At force magnitudes

$$F_{pl,2} = F_m = 4\ M_{pl} / R \qquad (6.8)$$

plastic hinges develop also at sections B. Since this instant the deformations grow faster and continue till the points A touch one another. Assuming (for simplicity) that the length of the arm R of the force F is approximately constant during this stage, the work performed from the beginning of plastic deforming till the maximum possible displacement R of the points A is achieved, is

$$L \approx F_{\mathrm{pl},2}\, R = 4\, M_{\mathrm{pl}} \,. \tag{6.9}$$

The plastic flow, and thus also the energy consumption, is concentrated in the regions of plastic hinges. The moments and stresses between them are lower. The energy consumption can be increased by creation of more plastic hinges. In general case of n radial forces, the elastic bending moments at sections A and B are:

$$M_{\mathrm{A}} = \tfrac{1}{2}\, FR\, (1/\varphi - \mathrm{cotg}\, \varphi) \;, \tag{6.10a}$$

$$M_{\mathrm{B}} = \tfrac{1}{2}\, FR\, (1/\varphi - 1/\sin \varphi) \,; \quad \varphi = \pi/n \,, \quad n \text{ is an integer.} \tag{6.10b}$$

The forces needed for full plastification (i.e. creation of plastic hinges also at sections B between the applications of point forces) are

$$F_{pl,n} = \frac{4}{\sin\varphi}\frac{M_{pl}}{R} = \frac{4}{\sin(\pi/n)}\frac{M_{pl}}{R} \tag{6.11}$$

REMARK. Euqation (6.7) holds if the ring thickness in the direction of its axis is small, so that the bending stresses at every point act only in the direction of tangent to the ring circumference. With large thicknesses (if, for example, the rings are created from tubes of length comparable or larger than the diameter) the stress distribution of is more complicated. Transverse load causes the plain strain state, where – in addition to the bending stresses in the circumferential direction – also stresses appear in the direction of the tube axis, of magnitude $\sigma_{\mathrm{y}} = \mu\sigma_{\mathrm{x}}$, where μ is the coefficient of lateral contraction (Poisson number). The resistance to plastic deforming is somewhat higher. This can be approximately respected by replacing the yield strength σ_{Y} in Equation (6.7) by the expression $\sigma_{\mathrm{Y}}\times(4/3)$.

The character of deforming can be influenced by the velocity of loading, especially at very high velocities, so that inertial forces apply. If several rings are connected together in the direction of the acting force, elastic impulse propagates through them at impact (see also Chapter 3). If the opposite end is fixed, the impulse is reflected with the same sign. The moments and stresses at this place are summed up, so that in addition to the plastic deforming of the first ring at the point of impact also the last ring at the place of support can be plasticised, while the rings between them are deformed not so intensively [2].

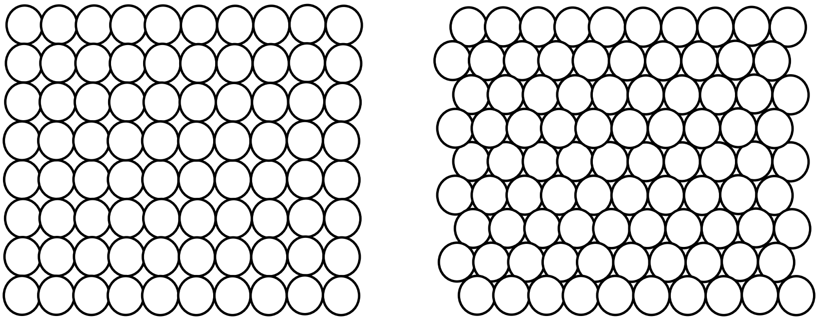


Figure 6.2. *Arrangement of rings: a) square, b - hexagonal*

Often, systems of connected rings are used. The character of their deforming is influenced by their arrangement. Figure 6.2*a* shows a square arrangement, and Fig. 6.2*b* shows hexagonal arrangement. In [2], Figure 4.19, it is shown that with the square arangement of rings the individual rings buckle, but with the hexagonal arrangement the individual rows buckle as a whole.

REMARK. Spring mattresses in old beds had also ring-like structure, and similar arrangement is also used in nets for retaining stones falling on a road from rocks.

6.2 Zig-zag structures

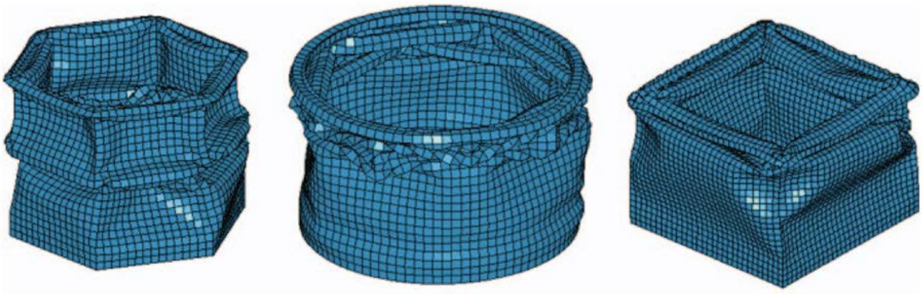
In some cases, metal bars bent in zig-zag way are useful. The load force can stretch or compress such „spring“; however, its sidesway buckling must be prevented. The bars are loaded by bending. The highest moment, acting at the folds, has the magnitude at the beginning of loading

$$M = F\,a\,. \tag{6.12}$$

Initially the structure behaves as an elastic spring. As soon as the moment attains the value M_Y according to Eq. (6.4), the material at the folds starts deforming plastically, and on attaining M_{pl} according to Eq. (6.7) plastic hinges are created here. With growing deformation the bending moment gradually increases to the value $M_{pl}\sqrt{2}$. The common metals usually strain-harden during larger deformations, so that, despite of the increase of the moment, the process proceeds in a stable manner till attaining the maximum compression, where the arms touch mutually along the whole length.

6.3 Axisymmetric shells loaded by axial force

With these shells, the local buckling of the walls appears at certain load. One example was given in Fig. 5.10c; three other are shown in Fig. 6.3.



Section: Hexagonal Circular Square

Figure 6.3. *Buckling of thin-walled tubes compressed by axial force [3, 4].*

These pictures were obtained by computer modelling; let us note that the finite element method (FEM) programs for nonlinear analysis of structures provide results conforming well to the results of experiments (compare the shape of local buckling in Fig. 6.3 with the photo in Fig. 5.10c). During the load increase further waves will arise at places yet unbuckled.

Cylindrical shells are used, for example, as absorbers of front impacts in cars or railway vehicles (Fig. 6.4). The absorbers sometimes have the shape of a truncated cone, with small or large angle [4]. They are made of metal or of fiber reinforced composites. Characteristic structural quantities (for the determination of critical load) are the modulus of elasticity, wall thickness and some other characteristic dimension, for example the diameter of a cylindrical or conical shell, or the length of a side in a square or hexagonal shell.

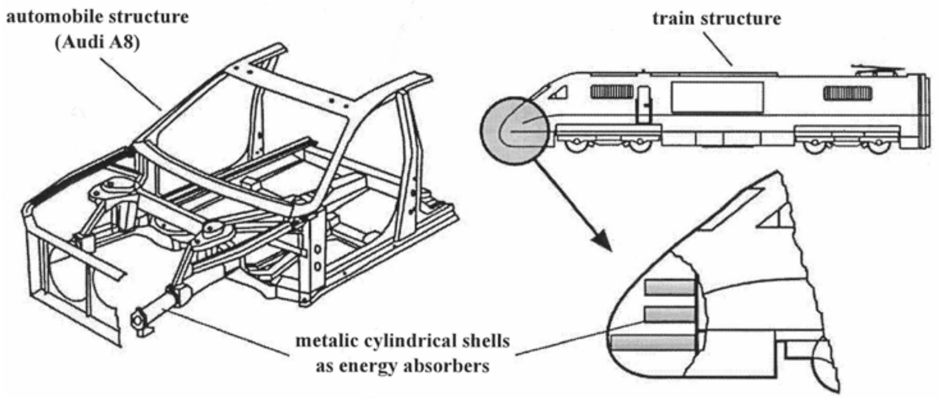


Fig. 6.4. *Absorbers of impact energy in cars and railway vehicles [5].*

6.4 Materials with cellular structure

The energy of impact can also be absorbed by irreversible deforming of **cellular structures** [2, 6, 7]. Important representatives are **honeycombs** and **foams**, known, for example, from sandwich structures (ski, corrugated board, walls of vehicles and aircrafts etc.). Honeycombs are plane configurations consisting of many cells, usually of the same form and size (Fig. 6.5). They are similar, in this respect, to the ring-like structures. Foams are spatial aggregates of cells resembling polyhedra. They can have valls closed or open, and their sizes vary. The material is metals, plastics or cardboard, but cellular structure is also typical of various plants, from grass to wood. They can be elastic, viscoelastic or elastic plastic, ductile or brittle.

An important geometry characteristics of cellular structures is the **relative density**,

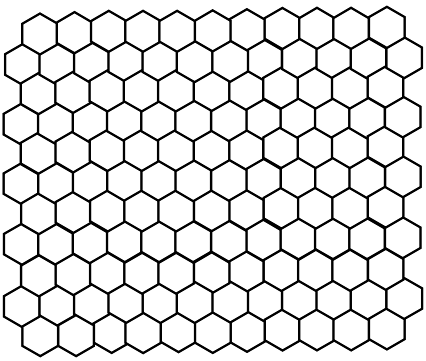


Figure 6.5. *Cells with honeycomb arrangement*

defined as $\boldsymbol{r} = \rho^*/\rho$, where ρ^* is the average density of cellular structure including the holes, and ρ is the density (kg/m^3) of cell material (e.g. polymer or metal). Relative density of common technical composites amounts several percent. The complementary quantity is **porosity** $\boldsymbol{\eta}$, expressing the proportion of pores in the composite; $\eta = 1 - \rho^*/\rho = 1 - r$.

Deforming of honeycombs or foams in damping of impacts resembles the deforming of ring systems. Typical stress-strain diagram is depicted in Figure 6.6; σ is the average macrostress calculated as the force acting in the honeycomb plane divided by the total cross section of the composite including the holes. It has three regions. Under low stress (region ***a***) the deformations are elastic and the stress increases in direct proportionality with deformation. At certain instant the stress attains the critical value. The thin walls in composites from elastic-plastic materials buckle and bend in elastic-plastic manner. Further compression occurs under constant or very slowly increasing load (plateau ***b*** in Figure 6.6). The walls collapse more and more. From certain instant, the initially opposite sides of the individual cells touch mutually, so that a compact body is formed, and a load needed to further compression grows quickly (region ***c***). Should the impact damping occur at approximately constant force, without peak values, the shock damper or energy absorber must be designed so that the assumed energy of impact is absorbed in region "b" in Figure 6.6.

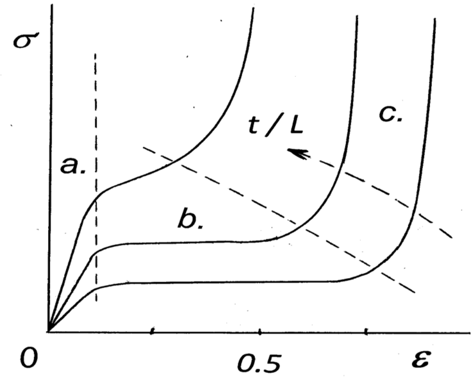


Fig. 6.6. *Stress-strain curves for compression of a honeycomb (after [2]).*
a – linear elasticity (cell walls bend), b – plateau (elastic buckling, plastic bending, brittle fracture), c – densification (cell walls are in contact).
L – charakteristic dimension of cell, t – cell wall thickness.

Elastic buckling of cell walls occurs at **critical load**, mentioned in Chapter 5.6. Some idea can be obtained from Euler´s theory of buckling of elastic bars, which also holds approximately for buckling of plates and walls in cells [8, 9]. The critical load for a straight bar loaded by axial force is

$$F_{cr} = \pi^2 E J /(c L^2) ; \qquad (6.13)$$

E is the tensile modulus of elasticity, L is the bar length, J is the moment of inertia of its cross section in bending, and c is a constant, characterising the support of the bar ends; for example, $c = 4$ if both ends are clamped. The moment of inertia of a rectangular cross section of width b and thickness t in the direction of bending is $J = bt^3/12$. The area of cross section is $S = bt$. Dividing the critical force by the area S gives the corresponding **critical stress** in the wall:

$$\sigma_{cr} = k E (t/L)^2 , \qquad (6.14)$$

where k is a constant. Critical is such stress, at which buckling should occur of an ideally straight bar, loaded by compressive force acting exactly in its axis. In reality, buckling occurs at significantly lower loads, due to various imperfections of the bar and load.

REMARK. Similarly to bars, the critical stress for local buckling depends on the modulus of elasticity E, wall thickness t, and a characteristic dimension L, for example the radius of a circular shell.

At buckling, the compressive stress is accompanied by bending stress, which can be many times higher, as it was shown in Chapter 5. Honeycombs or cells of elastic-plastic material deform plastically, with irreversible consumption of energy. If the walls are from a brittle material, they break on attaining the ultimite strength. The energy is consumed in fracture processes as well, though in lesser extent. But even these fractures, which occur at approximately constant stress, ensure that the compressive force is roughly constant.

The application of such structures for damping elements needs the knowledge of relationship between the force and compression for the intended kind of element.

6.5 Axisymmetrical tearing and inversion of metal tubes

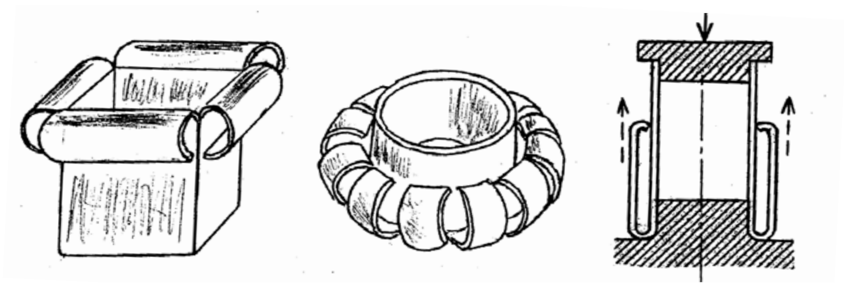
Creation of a plastic hinge consumes much more energy than elastic deforming.

A drawback in bending of a straight bar under transverse load is a limited extent of bending (only 90 – 180°) and, therefore, a limited amount of dissipated energy. Improvement can be reached by various ways. One of them [2] is **axisymmetrical tearing of tubes** in several directions simultaneously, for example by pressing a conical pin into them, followed by permanent bending achieved by winding the torn stripes. Figure 6.7a shows tearing of a tube of square cross-section: the pressing it onto a pyramidal pin on a massive support plate creates four wound stripes. Figure 6.7b shows simultaneous tearing of several stripes from a tube with circular cross section, pressed onto a suitably formed die. In both cases, suitable grooves can be made in the tube in advance for ensuring the demanded paths of fractures and constant width of created stripes. The total work W of the force F acting along the way $\boldsymbol{x}$, consists of the work of tearing, W_{tear}, work of plastic bending and winding of the stripes, W_{plast}, and work of friction W_{f} between the bent stripes and the shaping die they slide along:

$$W = F\,x = W_{\mathrm{tear}} + W_{\mathrm{plast}} + W_{\mathrm{f}}\,. \tag{6.15}$$

The stripes are either wound permanently, or again straightened after bending, depending on the shape of the forming die.

Another way for reshaping of relatively large extent is axisymmetrical bending with **inversion of tubes** from very ductile material, as shown in Fig. 6.7c [2]. If a suitably preformed tube is pressed onto a central stopper plate, the tube wall bends and is extended in the meridian direction and immediately bends back into the cylindrical shape of a larger diameter. The work of elastic deforming is negligible



a. b. c.

***Figure 6.7**. Forming of metal tubes: a, b) with tearing of walls, c) with inversion.*

compared to the work of plastic forming, which is

$$W = F\,x = W_{\text{plast,bend, 1}} + W_{\text{plast.bend, 2}} + W_{\text{membr}} + W_{\text{f}}\ , \qquad (6.16)$$

where $W_{\text{plast,bend,1}}$ is the work for meridian bending by 180° in one direction, and $W_{\text{plast,bend,2}}$ is the work of its straightening, W_{membr} is the work of membrane forces expended for permanent enlargement or reduction of the diameter at the place of forming, and W_{f} is the work of friction forces between the tube and the support.

The force depends on the deformation as follows. At the beginning the force necessary for deforming increases, but as soon as parts of the tube are inverted into new shape (tubular of a different diameter), the deforming continues under constant force. This means that a suitable preforming can create an element for impact damping, where the force is approximately constant from the beginning of work.

6.6 Composites

Impacts can be damped also by means of composite materials and parts. Two or more components with various properties are composed so that the resulting material has different (and better) properties. An example of a composite material is a laminate with a polymer matrix of relatively low strength and dispersed fibres of high strength, for example of glass, kevlar, or carbon. The laminates are shaped before curing, and sometimes after it they are machined to final dimensions. An example of a composite product is ski, created as a sandwich consisting of two strong elastic layers with a softer core from polymer foam between them.

A very wide range of properties of a composite can be achieved by the choice of components and their arrangement. If impacts should be mitigated, it is necessary that the composite has acceptable strength and especially high ability of absorbing the energy of impact. The applications are: parts of vehicles or aircrafts, protective helmets and shields, elements for protection of passengers during a collision, but also bullet-proof vests; they must prevent bullet penetration through the protective layer and, simultaneously, to reduce the maximum force caused by the projectile.

The consumption of energy during the destruction of **fiber reinforced composites** is influenced by the material of the fibres and of the matrix, by the properties of the interfaces between them, by the shape and fraction of fibres and also by their orientation, i.e. the angle of the fibers with respect to the direction of the load and axis of the body, and the order of laying the individual layers. A role is also played by the conditions of manufacturing, the component geometry and the conditions of use. If the properties depend on temperature, this factor is also important.

For the mitigation of impact effects the high strength is not so much important as the ability to absorb (or dissipate) the energy at appropriate (not too high) force or stress. Energy is absorbed by plastic deforming and also by fracture processes. The principles of fracture mechanics were explained in Chapter 4. The consumption of energy during fracture depends on specific fracture energy (i.e. energy needed for creation of fracture area of unit size) and on the area of the created fracture surfaces. The fracture of fibre composites can propagate across the fibres, in the matrix, or at the interface fibre – matrix. As the total area of the contact of matrix with many fibres is very large, it is advantageous if the fibres are strong and the interface between them and the matrix is less strong, and their arrangement is such as to promote the propagation of cracks and growth of fracture surfaces in the space between fibres. An example is intentional splaying of a composite during failure in some applications [2].

Sandwich parts can be used for catching impacts perpendicular to the surface. From macroscopic point of view, they are plates. The energy is accumulated and absorbed by bending, extension and fracture of outer layers, and by localized crushing of the core between them. Honeycombs and foams are very suitable for sandwich cores. The core can delaminate from the outer layers and sometimes a small and sharp body can break through the sandwich [2]. A role is sometimes played by stress waves and, perhaps, their reflections at the end opposite to impact.

A special kind of sandwich is corrugated cardboard from paper or other material, used in packaging technology for protection of goods during manipulation and transport. The core, whose arrangement is similar to one or more rows of cells in Figure 6.5, is deformed under high forces; its walls warp and dampen the impact.

Specific applications of composites are **bulletproof vests** [10]. Here, the experience is utilized that a bullet (from a hand gun) can be slowed down in an efficient way by several layers of strong fabric or fibers. Very strong and tough fibres, for example of Kevlar, "catch" the projectile and blunt its sharp tip (if it is from lead) and spread the effect onto a larger area from a layer to layer. This reduces the load concentration. In some applications the fibres are coated by a resin, and then several these vowen or laminated layers are stacked and inserted between two layers of polyethylene. Even more layers can be created. Another problem a bulletproof vest must face is the high momentum (product of the mass and velocity) of the bullet or debris (e.g. from a grenade), so that wounding could be caused not by the sharp tip, but to high force acting over a large area. Therefore, various plates are sometimes inserted into the protective structure, made either of metals, for example steel or titanium, or from ceramics of very high strength and fracture toughness, e.g. Al_2O_3 or from carbides of some metals. The plates, which can reduce the momentum of the bullet and make it blunt and distribute the load and energy on a larger area, are suitable because they do not limit the mobility of the user in contrast to continuous armor. For example, similar kind of protection is used by some animals, for example armadillo, whose armor is made of hard chitin shell plates connected by leather. Certain drawback of metal inserts is higher mass (compared to textile), and brittleness of ceramic plates, so that such protective vest is unable to catch multiple hits at the same place.

6.7 Air- or gas bags and cushions

We have seen several times that an impact can be mitigated if an elastic stop is used, for example by a metal spring. Similar servis can also be done by an air spring, or filled by another gas. Well known are also bubble foils used for packaging of goods, or airbags used in cars.

Airbag works as follows. A textile bag is hidden at a suitable place (for example, in the steering wheel or in a dashboard in front of the co-driver or passenger. An impact activates a mechanism, which very quickly (usually by a pyrotechnic process) releases the bag from the cover and triggers the gas generator, using sodium azide (NaN_3) or other suitable chemical. The created gas (nitrogen in this case) fills the bag in front of the protected person that leans on it. The filled bag dampens the impact and distributes the acting force uniformly over a larger area. The design solution can be such that the gas filling slowly escapes through the pores in the fabric. This helps to prevent a too high peak force at the maximum compression of the bag. (Braking with constant force would be ideal.)

Small air bags are often used in packaging technology for protection of goods transported in hard boxes. They can be in form of **bubble foils** with many small bubbles (cm in size) or sealed thin plastic bags filled with air. They are cheap and very light, and their shape is adjusted easily to the space available or to the gap between the goods and the stiff container.

The next section will summarise the principal formulae for the gas cushion. According to the equation of state of ideal gas, the following relationship holds between the gas pressure p and volume V of this gas [11, 12]:

$$p\,V^{n} = p_0\,V_0^{n}\ . \qquad (6.17)$$

Subscript 0 denotes the values pertaining to the beginning of compression, and the exponent n is a constant, characterizing the process, which is, generally, polytropic. For isothermic change, $n = 1$, for adiabatic process (without heat exchange with the surroundings) $n = 1.4$. The initial pressure p_0 can be the same or higher than the atmospheric pressure, depending on the construction of the particular damping element. The volume of the gas filling in the cushion is constant, and if also the area S of the contact of the decelerated body and the cushion will be constant, the simple relationship exists between the pressure p and the cushion height H:

$$p\,H^{n} = p_0\,H_0^{n}\ , \quad \text{resp.} \quad p / p_0 = (H_0 / H)^{n}\ . \qquad (6.18)$$

REMARK. If a bubble foil is used, the total area S equals the sum of the loaded bubbles.

We also want to know the relationship between the pressure p and compression δ of the cussion. As it holds $\delta = H_0 - H$, Equation (6.18) can be rewritten as follows:

$$p / p_0 = [H_0/(H_0 - \delta)]^{n} = [1 - (\delta/H_0]^{-n}\ , \quad \text{resp.} \quad p = p_0\,[1 - (\delta/H_0]^{-n} \qquad (6.19)$$

For example, the compression of the initial cushion height by 10% causes (with n = 1.4) the pressure increase by 16%; compression by 20% (i.e. to 80 % of the initial height) increases the pressure by 37 %, etc. With constant contact area, simple relationship between the pressure and force holds:

$$F = p\,S\ . \qquad (6.20)$$

REMARK. The pressure p_0 in the unloaded damper is not observable for anybody out of the system.

If a gas cushion is used for impact damping, we are interested in the pressure increment Δp during the compression from zero (or equilibrium) position instead of the pressure p in the closed cushion. Only this increment creates the braking force. From the relationship

$$p = p_0 + \Delta p\ . \tag{6.21}$$

where p_0 denotes the pressure in unloaded damper, we obtain

$$\Delta p = p - p_0 \tag{6.22}$$

After expressing the pressure p from Equation (6.19) we get after a rearrangement

$$\Delta p = p_0 \left\{ \left[1 - (\delta / H_0) \right]^{-n} - 1 \right\} \tag{6.23}$$

The energy of gas increases during the compression from the volume V_0 to V [8, 9]:

$$E_{pot} = p_0 V_0^{\ n} \int_{V_0}^{V} V^{-n} dV = \frac{p_0 V_0}{n-1} \left[1 - \left(\frac{p}{p_0} \right)^{\frac{n-1}{n}} \right] ; \tag{6.24}$$

V_0 and V are the volumes of the cushion before and after compression, and p_0, p are the corresponding pressures. In common cases atmospheric pressure can be used for p_0. For a fast event, exponent n = 1.4, usual for adiabatic process, may be used, or somewhat lower. The initial volume of the cushion of cylindrical shape with area S and height H_0 is

$$V_0 = S\, H_0 \tag{6.25}$$

Energy E_{pot}, given by Equation (6.24), was taken from the stopped body, and is only accumulated in the compressed gas. As soon as the maximum compression was attained, the stopped body would be returned back. However, suitable design can ensure that the gas will be drained out at the instant of stopping or very shortly after it, so that no rebounding occurs.

Solution of equations (6.22) – (6.24) for design parameters p_0, V_0, S and energy E_{pot} that should be accumulated (or dissipated) can give the volume in compressed state and the corresponding maximum pressure and braking force. Also it is possible to determine the cushion volume for the given energy and maximum permitted force.

6.8 Hydraulic shock absorbers with constant force

In the previous chapter the stopping was investigated for various parameters of the braking device (a spring with linear characteristics, friction or viscous damping, etc.). Here, the case will be investigated with constant deceleration, ensured by a hydraulic damper. In this damper a moving piston presses the working fluid (oil, for example) through one or more openings of smaller cross section. Due to throttling effect, overpressure arises at the openings, which acts against the piston and causes the braking. Common vibration dampers use constant cross section of the openings. Since the pressure difference on a slot or opening is higher for higher velocity of the liquid, the braking force increases with the piston velocity. Such kind of damper is not suitable for braking or shock damping, as it gives nonlinear course of braking force with maximum value at the beginning of braking (Fig. 4.4).

Constant deceleration is achieved if the cross section area of the throttling slot gets smaller during braking in such way, that even during the decreasing velocity of the piston the pressure of the working liquid is constant.

Reduction of the cross section area can be either continuous (for example, the piston closes a specially shaped opening in the cylinder wall, Fig. 6.8a, or a specially shaped needle closes an opening in the front of the cylinder, Fig. 6.8b), or successive, where several slots in the cylinder wall are gradually closed by the moving piston (Fig. 6.8c) so that the area for the liquid flow becomes smaller. In the last case (Fig. 6.8c) the deceleration and braking force are not constant, but vary as the individual slots are successively closed. In the following paragraph the

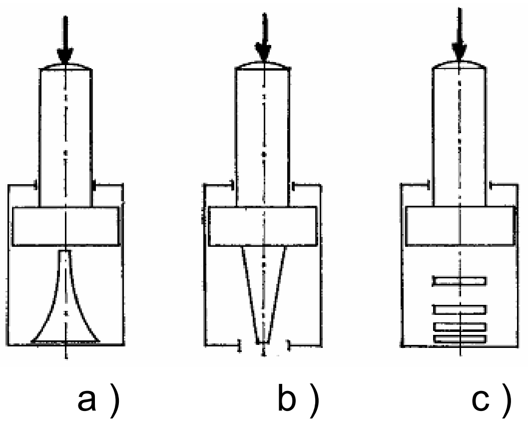


Fig. 6.8. *Hydraulic shock absorbers with constant deceleration.*
a) variable size of the opening in the wall, b) a needle with variable diameter, c) series of slots of various distances in the wall.

arrangement according Figure 6.8a will be addressed.

Continuous closing of the throttling opening

If the throttling opening is short (e.g. in the bottom of the cylinder), turbulent flow arises here, and the pressure difference and also the pressure on the piston, is [13]:

$$p = \frac{\xi \rho v_k^{\,2}}{2} \;; \qquad (6.26)$$

v_k – flow velocity,
ρ – density of the liquid,

ξ – loss factor; it depends on the viscosity, shape of the opening, flow rate and other factors. It is approximately constant for certain shock absorber.

The velocity v_k of the liquid in Eq. (6.26) can be expressed by piston velocity v as

$$v_k = v\, S_p / S \;; \qquad (6.27)$$

S_p is the effective piston area, and S is the cross section area of the throttling opening. Considering that the piston velocity, cross section area of the opening, and also the pressure p can depend on the piston position x, one obtains

$$p = \frac{\xi \rho S_p^{\,2}}{2} \left(\frac{v(x)}{S(x)} \right)^2 . \qquad (6.28)$$

If the deceleration should be constant, a_b = const., it must – with respect to formula

$$a = \frac{d^2 x}{dt^2} = \frac{d(v^2)}{2dx} = -a_b \qquad (6.29)$$

hold for the piston velocity:

$$v(x) = \sqrt{v_0^{\,2} - 2 a_b x} = v_0 \sqrt{1 - (x / x_b)} \;, \qquad (6.30)$$

where v_0 is the velocity at the beginning of braking, and x_b is the braking path.

After expressing the braking force F_b by means of mass m and deceleration a_b of the body, and pressure p of the liquid and the piston area S_p,

$$F_b = m\, a_b = -p\, S_p \;, \qquad (6.31)$$

one obtains, after combination of Eqs (6.27) – (6.31), the following relationship of the cross section area of the opening and the piston position x [14, 15]:

$$S(x) = \sqrt{\frac{\xi \rho S_p{}^3 x_b}{m}\left(1 - \frac{x}{x_b}\right)} = S_0 \sqrt{1 - \frac{x}{x_b}} \ , \qquad (6.32)$$

where

$$S_0 = \sqrt{\xi \rho S_p{}^3 \, x_b / m} \qquad (6.33)$$

is the total cross section of the opening at the beginning of braking. (The course of relative change of the cross section area, $S(x)/S_0$, expressed by means of relative path of braking, x/x_b, is the same for every damper with constant deceleration!)

Equations (6.27) – (6.33), together with detailed analysis, reveal the following features of this kind of dampers:

1. A body of mass m and initial velocity v_0 will be stopped on the path x_b at constant deceleration

$$a_b = \frac{v_0{}^2}{2x_b} \ ; \qquad (6.34)$$

the braking force is given by Eq. (6.31). Energy dissipated during braking is

$$W = \frac{m v_0{}^2}{2} \ . \qquad (6.35)$$

2. If the actual mass m' of the decelerated body is higher than that assumed mass m, the shock damper is not able to decelerate this mass sufficiently. The velocity decreases more slowly, so that the actual velocity at certain place is higher than the assumed one, and the braking force is also higher. The relative increase of the braking force grows during braking. On the other hand, with relatively small mass the velocity decreases faster, and the braking force is lower than initially assumed; however, the duration of stopping increases. The influence of changes of the mass can be eliminated by the change of the cross section of the throttling opening so as the equations (6.32) and (6.33) are fulfilled.

3. If the actual initial velocity v_0' of the stopped body is higher than the initially assumed, v_0, the course of velocity during braking will be similar, and the actual velocities will be all the time higher in the ratio v_0'/v_0; the duration of stopping will be shorter in the same ratio. The deceleration and braking forces will be higher in

the ratio $(v_0'/v_0)^2$. The influence of the change of initial velocity can be eliminated by the change of braking path so that the relationship (6.30) is retained.

Hydraulic shock absorbers can be used repeatedly. Therefore they are used in various machines or appliances for manipulation, especially if the motion of heavy objects should be damped, for example the moulds in glass forming machines. In such cases it is necessary to monitor the dissipated power and check whether the generated heat can be transferred away in a natural way, or to propose suitable cooling of the working liquid. The mean power of the damper is calculated as

$$P_{stř} = \frac{energy\ dissipated\ in\ one\ work\ cycle}{cycle\ duration} \tag{6.36}$$

The theory of hydraulic impact absorbers with constant deceleration is described in more detail in [14, 15], and also the absorber with a series of slots in the cylinder wall, which are gradually closed by the piston. Information on some commercial hydraulic shock absorbers can be found, for example, in [16, 17].

Further commercial impact dampers and energy obsorbers are described in [18] – [20]. Today, computer simulations of demanding problems are used more and more. For example, information on programs for simulations of collisions (e.g. at crash tests) can be found in [21 – 23], some videosimulations are in [24 – 26]. (All these cases correspond to the situation on the web as of May 2018.)

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7. Vibrations and mitigation of their effects

Rotational and other periodical movements are sources of parasitic loads in motors, machines and other devices. In imperfectly balanced equipment it is manifested by vibrations and forces transmitted to the foundations or adjacent bodies. Here it will be shown what these vibrations and forces depend on and how they can be mitigated. At first, basic concepts will be reminded; more details can be found in [1 – 7]. Vibration of bodies is possible thanks to two properties: elasticity and inertia. If an elastic body is loaded, it is deformed. After the load is released, it returns to the initial state. Due to inertia, overshooting in the opposite direction usually follows, again a return, etc. – the body starts vibrating. If no external forces act, one speaks of free vibrations. However, such movement is usually damped and ceases after a while. Without damping, oscillatory movement would continue without any limitations. Often, the vibrations are forced – for example by a motor or by movement of the equipment.

In this chapter, free undamped and damped vibrations will be discussed first (Chapters 7.1 and 7.2). Chapter 7.3 is devoted to forced vibrations, and Chapter 7.4 deals with the transmission of forces from the vibrating body to the foundations. Then, kinematic excitation will be explained, which sometimes appears in transport means. Chapters 7.6, 7.7 and 7.8 are devoted to the transverse vibrations of beams and shafts, circular vibrations, and to the case where the centre of gravity lies not at the axis of rotation, but has some eccentricity. Problems of balancing are also mentioned briefly. Chapter 7.9 is devoted to energies in vibrating systems, chapter 7.10 explains the function of dynamic absorbers of vibrations, and chapter 7.11 explains briefly vibrations in systems with several degrees of freedom.

7.1 Free vibration without damping

The simplest case is represented by a body joined with a spring (Fig. 7.1). The word spring denotes here any elastic body, which starts vibrating after being deflected from undeformed state and released. Various beams and shafts belong to this category. The vibration is called free if no external force acts on the body.

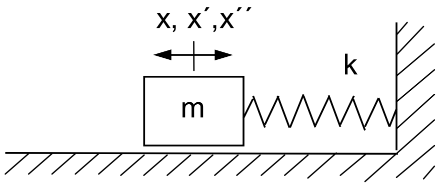

***Figure 7.1**. Free vibration.*

If the body is small compared to the spring dimensions, and its mass is big, we can speak of a mass point. The corresponding motion equation is:

$$m\ddot{x} + kx = 0 \ ; \qquad (7.1)$$

m is the mass of the body (kg), x is its path (m), and k is the spring stiffness, expressing the force needed for unit deformation (N/m). The dot above the symbol denotes the derivative with respect to time, dx/dt, two dots mean the second derivative, d^2x/dt^2. Division of Eq. (7.1) by the mass m gives differential equation:

$$\ddot{x} + \omega^2 x = 0 \ ; \qquad (7.2)$$

ω is the **natural circular frequency** (angular velocity) of the vibrating movement (s^{-1}), determined as

$$\omega = \sqrt{\frac{k}{m}} \ , \qquad (7.3)$$

and related with the frequency of vibrations f as follows

$$\omega = 2\pi f \ . \qquad (7.4)$$

The solution of Equation (7.2) is:

$$x(t) = A \sin(\omega t) + B \cos(\omega t) \ , \quad \text{resp.} \quad x(t) = C \sin(\omega t + \varphi_0) \ . \qquad (7.5)$$

A and B, or C and φ_0 are constants that can be determined from initial conditions. C means the amplitude of vibrations, and φ_0 is the angle corresponding to the position of the material point at time $t = 0$.

REMARK. The spring was considered as immaterial. In fact, it has also some mass, which influences the natural frequency of the system. This influence can be considered (in the determination of natural frequency) by adding certain part of this mass to the mass of the vibrating body; for example one third for a helical spring.

7.2 Free vibration with damping

Motion is often hindered by some resistance, for example by friction or viscous damping. Here we shall look at vibrating movement with damping force proportional to the velocity (Figure 7.2), as usuall with hydraulic dampers. The motion equation is

$$m\ddot{x} + b\dot{x} + kx = 0 \,. \tag{7.6}$$

Constant b (Ns/m) expresses the resistence corresponding to unit velocity. Division of Eq. (7.6) by the mass m gives the equation

$$\ddot{x} + 2N\dot{x} + \omega^2 x = 0 \,; \tag{7.7}$$

ω is the circular frequency, defined in Equation (7.3), and

$$N = b/(2m) \tag{7.8}$$

is the **coeficient of damping** (s^{-1}). Number 2 in the denominator is for formal reason.

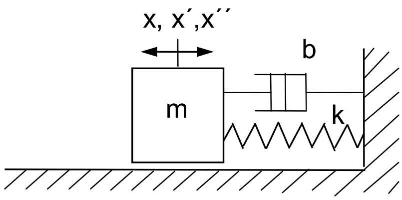

Figure 7.2. *Free vibration with damping.*

The form of solution of differential equation (7.6) depends on the magnitude of constants m, b and k, and thus also on the damping. The solutions of this equation for three various cases were described in Chapter (4.2). Here, undercritical damping will be discussed first, where the movement can be described as follows

$$x(t) = C\,e^{-Nt} \sin(\omega_1 t + \varphi_0) \,. \tag{7.9}$$

It is also oscillatory movement, but with the amplitude decreasing in exponential way (Fig. 7.3). Circular frequency of the damped system is

$$\omega_1 = \sqrt{\omega^2 - N^2} = \omega\sqrt{1-\delta^2}\,, \tag{7.10}$$

where

$$\delta = \frac{N}{\omega} \tag{7.11}$$

is **damping ratio** (relative damping). Vibration for $\delta < 1$ is thus somewhat slower than undamped vibration for the same parameters m and k, more for higher ratio δ. The case $\delta = 1$ corresponds to critical damping, where the deflected body returns only to the initial position. $\delta > 1$ pertains to overdamping, also without vibration, but with faster attenuation.

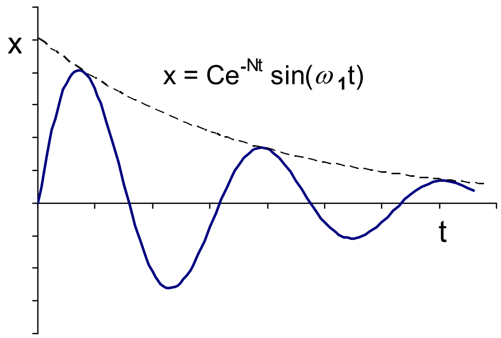

Figure 7.3*. Free vibration with damping.*

7.3 Forced vibration

This is a very common case. The pricipal features will be shown on an elastic, or elastically supported body, exposed to the action of harmonic force $F_0 \sin(\Omega t)$. Also damping proportional to the velocity is present (Fig. 7.4). Ω is the circular frequency of the driving force (= excitation frequency). The motion equation is

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin(\Omega t). \tag{7.12}$$

Its division by the mass m yields the following equation for the displacement:

$$\ddot{x} + 2N\dot{x} + \omega^2 x = a_p \sin(\Omega t + \varphi_p)\,, \tag{7.13}$$

where

$$a_p = F_0/m \tag{7.14}$$

is the amplitude of acceleration in forced vibrations, and ω is the natural circular frequency of the free undamped vibrations, given by Equation (7.3). Differential

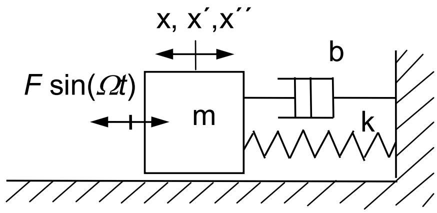

Figure 7.4. *Forced vibration.*

Equation (7.13) is nonhomogeneous (with right side). Its solution is obtained as a sum of the solution x_h of homogeneous equation (7.7) without right side, and the so-called particular integral x_p:

$$x = x_h + x_p \,. \tag{7.15}$$

The solution of homogeneous equation is the same as for free vibration. For $\delta<1$,

$$x_h(t) = C\,e^{-Nt} \sin(\omega_1 t + \varphi_0) \,. \tag{7.16}$$

Particular integral x_p is a function that conforms to Eq. (7.13). A suitable expression is

$$x_p(t) = r \sin(\Omega t + \varphi_p) \,. \tag{7.17}$$

After inserting Equation (7.17) and its derivatives into (7.13) one obtains [1]:

$$r = a_p \frac{1}{\sqrt{(\omega^2 - \Omega^2)^2 + (2N\Omega)^2}} = \frac{a_p}{\omega^2} \frac{1}{\sqrt{(1-\eta^2)^2 + (2\delta\eta)^2}} \,, \tag{7.18}$$

$$\varphi_p = arctg \frac{b\Omega}{k - m\Omega^2} \,. \tag{7.19}$$

In these expressions, damping ratio δ, defined by Eq. (7.11), is used and also **tuning factor** η, defined as the ratio of exciting frequency to the natural frequency of the undamped system:

$$\eta = \Omega / \omega \,. \tag{7.20}$$

If the exciting frequency Ω is the same as natural frequency ω, the system is in resonance.

The resultant complete solution of Equation (7.13) is

$$x(t) = C\,e^{-Nt} \sin(\omega_1 t + \varphi_0) + r \sin(\Omega t + \varphi_p) \ , \qquad (7.21)$$

where C is a constant. Regardless the intensity of damping, the first component (with the term $e^{-N\,t}$), representing transient phenomena, disappears during some time. Then, only the stationary component remains

$$x(t) = r \sin(\Omega t + \varphi_p) \ , \qquad (7.22)$$

which describes the forced stationary vibrations. These are usually decisive for behaviour of machinery appliances. Amplitude r and phase shift φ_p of stationary component are important for the evaluation of forced vibrations. It is useful to introduce the **relative amplitude** $\boldsymbol{\xi}$ as the ratio of the amplitude r and static deformation r_{st} corresponding to the amplitude F_0 of exciting force:

$$\xi = r/r_{st} \ , \text{kde} \quad r_{st} = F_0/k \ . \qquad (7.23)$$

It can be proven that

$$\xi = \frac{1}{\sqrt{(1-\eta^2)^2 + (2\delta\eta)^2}} \ , \qquad (7.24)$$

where δ is relative damping. Another useful quantity is **phase shift** $\boldsymbol{\varphi}$, expressing the delay between the exciting force and the deflection of the vibrating mass:

$$\varphi = \varphi_P - \varphi_x = \text{arctg} \frac{2\delta\eta}{1-\eta^2} \ . \qquad (7.25)$$

Both quantities, ξ and φ, are functions of tuning η and relative damping δ. Their courses are depicted in Figures 7.5 and 7.6. The dependence of the amplitude of forced vibrations on frequency, Fig. 7.5, is called **amplitude charakteristics** or **resonance curve**, and the dependence of phase shift on η, Fig. 7.6, is called **phase charakteristics**. Both diagrams illustrate well the importance of resonant frequency. We see that if the exciting frequency approaches to the resonant frequency ($\Omega \to \omega$, resp. $\eta \to 1$), the amplitude of non-damped vibration ($\delta = 0$) grows to infinity. Equation (7.24), adapted for non-damped vibration, changes to:

$$\xi = 1/(1 - \eta^2)\,, \qquad \text{tg}\,\varphi = 0 \;\; (\text{pro}\;\eta \neq 1) \tag{7.26}$$

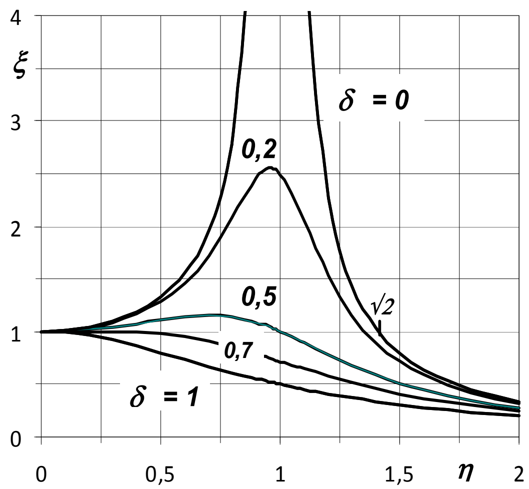

Figure 7.5. *Resonance curve (relative amplitude ξ of forced vibration as a function of the ratio η of exciting frequency Ω and natural frequency ω). ξ – relat.ampl.(= r/r_s,), η – tuning factor (= Ω/ω), δ – relative damping (= N/ω).*

With η = 1, resp. at $\Omega = \omega$, one speaks of resonancy, and $\xi \to \infty$. However, infinitely large amplitude is only a hypothetical case. In reality, beginning from certain magnitude, deviations from linear dependency of deflection on force appear. On the other hand, intensive vibrations can make a proper operation of the object impossible, or even cause its damage or destruction, tearing-off of the foundation and other damages. They are thus very dangerous, and effort must be made so that the operation revolutions or frequency of a machine are sufficiently far from the resonant ones. Fortunately, even if the true revolutions were at certain instant the same as the resonant ones, the destruction does not occur instantaneously. If Equation (7.13) is adapted for non-damped case, that is

$$\ddot{x} + \omega^2 x = a_p \sin(\Omega t + \varphi_p)\,, \tag{7.27}$$

the particular integral for $\Omega = \omega$ has the shape

$$x_p(t) = r\, t \sin(\Omega t + \varphi_x)\,, \tag{7.28}$$

where

$$r = a_p/(2\omega)\ , \quad \varphi_x = \varphi_p - \pi/2\ . \qquad (7.29)$$

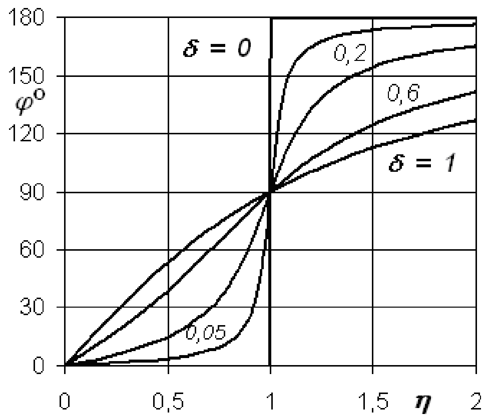

Figure 7.6. *Phase characteristics (phase shift φ of forced vibration as the function of the ratio η of exciting frequency Ω and natural frequency ω). φ – phase shift (^{o}), η – tuning factor, δ – relative damping*

Euqation (7.28) says that in the resonance and with zero damping the amplitude increases proportionally with time t (Fig. 7.7). The operation revolutions or frequencies are often higher than the resonance ones. During the start of the machine in such case it is necessary to pass through the resonant speed as quickly as possible, so that the amplitude has not enough time to attain dangerous magnitude.

REMARK. Everybody, who has observed an electric table grinder, has surely noticed that at certain instant after being switched-off, the grinding machine vibrates strongly for a short while. This is just when passing through the resonant frequency. After a while this behaviour ends, as the revolutions have dropped sufficiently away from the resonance.

During resonance of an appliance without damping also the phase of vibrations changes suddenly in an ideal case (Fig. 7.6). This is obvious from Eq. (7.25). The angle φ is positive for $\eta < 1$, but negative for $\eta > 1$. The relative amplitude ξ for $\eta > 1$ thus should be plotted in the negative half-plane, i.e. below the horizontal axis. Because of a uniform way of plotting undamped and damped vibrations, also this case ξ is usually plotted in the positive half-plane.

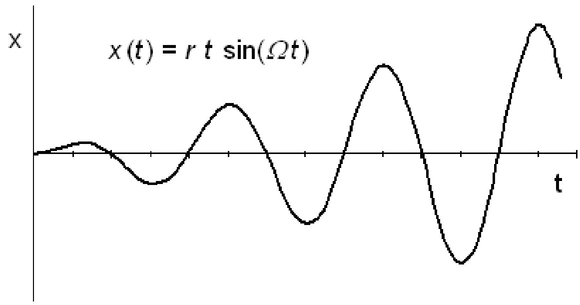

***Figure** 7.7. Gradual growth of vibrations at resonance.*

If damping is present in a body excited by a harmonic force, the amplitude during resonance will have only limited magnitude, and no change of phase occurs. With increasing damping, the amplitude becomes smaller, and the resonance occurs at lower exciting frequency, as it is obvious from Figure 7.5. The position of the peak of resonance curve can be found from the condition $d\xi/d\eta = 0$. If Equation (7.24) is differentiated with respect to η and the obtained expression is put equal zero, the following value of the tuning factor is obtained

$$\eta_p = \sqrt{1-2\delta^2} \ . \tag{7.30}$$

Subscript „p“ means that it corresponds to the peak of the resonance curve. It follows from Eq. (7.30) that for relative damping $\delta = 1/\sqrt{2}$ it holds $\eta_p = 0$, which means that the maximum lies in the origin of the coordinate system ($\Omega = 0$). With more intensive damping ($\delta > 1/\sqrt{2} = 0.7071$), the resonance curve has no maximum, and the relative amplitude decreases continuously with increasing velocity. It holds that for $\delta > 1/\sqrt{2}$ the amplitude of forced vibrations is always smaller than the deflection caused by static force of the same magnitude.

The maximum relative deflection is

$$\xi_{max} = 1/[2\delta\sqrt{(1-\delta^2)}] \quad , \tag{7.31}$$

which can be expressed for small damping approximately as

$$\xi_{max} \approx 1/(2\delta) . \tag{7.32}$$

If the movement is damped in some way, the phase angle φ between the exciting force and deflection is smaller than 180° for any frequency (Fig. 7.6). For $\eta < 1$ it is $\varphi < \pi/2$, for $\eta > 1$ it is $\varphi > \pi/2$. At resonance, $\varphi = \pi/2$.

Let us look at one important thing in Fig. 7.5. The relative amplitude ξ below resonance for undamped movement ($\delta = 0$) is always larger than 1. Above the resonance, $\xi > 1$ only for the tuning factor $\eta < \sqrt{2}$. The absolute value of the relative amplitude for $\eta > \sqrt{2}$ is always lower than 1; this means that dynamic deflections will be smaller than the static ones. This is used in practice for reduction of deflections of vibrating bodies – it is sufficient if the operation revolutions of the machine are significantly higher than the resonant ones. The damping is not decisive in this case. However, damping helps to make the dynamic deflections smaller – but only them. It does not pertain to the transmission of forces to other parts, where its influence is negative, as it will be shown in the next part.

7.4 Forces transmitted from the vibrating body to the frame or foundations

It is obvious from Figure 7.2 that the forces of the spring (S) and the damper (D) are transmitted to the foundations:

$$R = S + D = kx(t) + b\dot{x}(t) . \tag{7.33}$$

Both components vary with time. Considering only the stationary case, one obtains

$$x(t) = r \sin(\Omega t + \varphi_x) , \quad \dot{x}(t) = \Omega r \cos(\Omega t + \varphi_x) . \tag{7.34}$$

The total force, transmitted into the foundation, is

$$R(t) = R_0 \sin(\Omega t + \varphi_R) , \tag{7.35}$$

where

$$R_0 = r\sqrt{k^2 + b^2\Omega^2} , \qquad \varphi_R = \mathrm{arctg}\,(c\Omega/k) = \mathrm{arctg}\,(2\delta\eta) . \tag{7.36}$$

The amplitude of force transmitted into the frame or foundation can be rewritten as

$$R_0 = F_0 \, \frac{\sqrt{1+(2\eta\delta)^2}}{\sqrt{(1-\eta^2)^2+(2\eta\delta)^2}} \tag{7.37}$$

The ratio of amplitude of the force transmitted to the frame and the excitation force is called **transmission coefficient** or **transmission ratio** $\boldsymbol{\xi_R}$,

$$\xi_\mathrm{R} = R_0/F_0 \ , \tag{7.38}$$

and can be written as follows:

$$\xi_\mathrm{R} = \frac{\sqrt{1+(2\eta\delta)^2}}{\sqrt{(1-\eta^2)^2+(2\eta\delta)^2}} \ . \tag{7.39}$$

The dependence of the transmission coefficient on the excitation frequency or tuning coefficient is shown in Figure 7.8. All curves pass through the coordinates $\xi_\mathrm{R} = 1$, $\eta = \sqrt{2}$, regardless the magnitude δ of damping. It is obvious from the picture that the elastic mounting of the vibrating mass reduces the force transmitted into the frame or foundation only if the exciting frequency is sufficiently high above the resonant frequency, that is for $\eta > \sqrt{2}$. We also see that for $\eta > \sqrt{2}$ the damping makes the situaton worse, more for stronger damping. If the equipment will work sufficiently far from resonance, the best solution is without any damping.

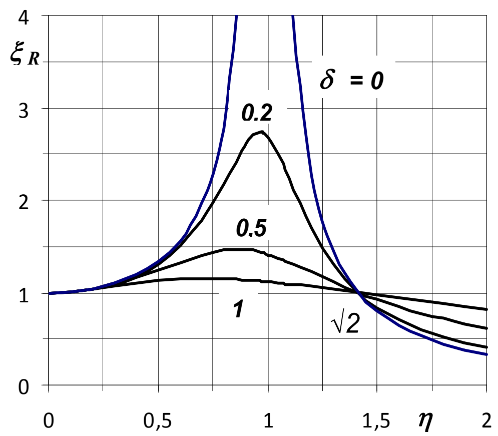

Figure 7.8. *Transmission coefficient ξ_R (= R_0/F_0) as function of exciting frequency. η – tuning coefficient (= Ω/ω), δ – relative damping (=N/ω).*

A practical conclusion from this section is:

Minimisation of forces transmitted from a periodically working appliance into a frame or foundations is best if the exciting frequency is significantly higher than the natural frequency of this equipment.

However, also a different solution exists, namely dynamic absorber of vibrations, which will be explained in Chapter 7.10.

7.5 Vibration caused by kinematic excitation

Until now, we assumed that vibrations are caused by harmonic force acting on an elastically supported body. However, a case can exist where a body is hanged on a spring, and periodic movement is enforced to the opposite end of this spring. Vehicles, going on a wavy road, belong to this category. Figure 7.9 shows the situation for a vehicle part pertaining to one wheel. The mass *m* rests on a spring, whose lower end is connected with a wheel (of zero mass, for simplicity), which goes on a wavy road. The mass had a tendency to vibrate by its own frequency *f*, but it is enforced to vibrate by the excitation frequency of the road. If the waviness has sinus-like course, the wheel makes (after the attenuation of transient phenomena) harmonic movement in vertical direction

$$y_1 = a \sin(\Omega t) \quad ; \tag{7.40}$$

a is the height of a half-wave, *t* is time, and Ω is the angular velocity, related to the velocity *v* of the ride by the relationship

$$\Omega = \pi v / L \, ; \tag{7.41}$$

L is the length of the half-wave of the unevenness (Fig. 7.9). The spring acts on the

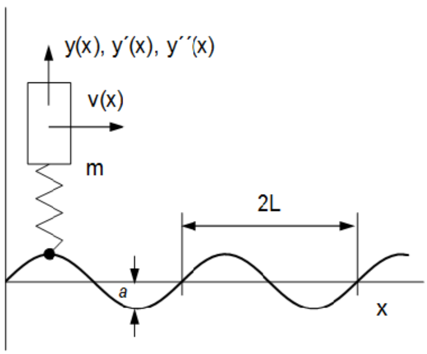

Figure 7.9. *Kinematic exciting. A vehicle going on a wavy road.*

body m by the force proportional to its compression $y - y_1$, where y is vertical displacement of the body from equilibrium position. The motion equation of the body m (with respect to the absolute space x, y) is

$$m\ddot{y} + k(y - y_1) = 0\ . \tag{7.42}$$

If y_1 is expressed from Eq. (7.40), one obtains

$$\ddot{y} + \omega^2 y = \omega^2 a \sin(\Omega t)\ ; \tag{7.43}$$

$\omega^2 = k/m$ is the square of circular frequency of natural vibrations of the mass m on the spring k. Equation (7.43) is similar to Equation (7.27) and also has a similar solution. The magnitude of deflection of the mass from equilibrium position is

$$y = a\,\frac{1}{1 - \dfrac{\Omega^2}{\omega^2}}\sin(\Omega t) = y_0 \sin(\omega t) \tag{7.44}$$

y_0 is the amplitude of vertical movement in the absolute space.

Equation (7.44) says that for very low velocities of a ride, and thus for low ratios of Ω/ω, the motion of the mass m copies the road unevennesses. For exciting frequency Ω, approaching the natural frequency ω of the body on a spring (= resonancy), the body will vibrate significantly. At high velocities, i.e. at $\Omega >> \omega$, the amplitude of vibrations will approach to zero. The body is unable to react to quick changes and goes on the road relatively smoothly. However, it vibrates in the opposite phase to the exciting force. An example for illustration follows.

Example.
A wheel goes on a wavy horizontal road by constant velocity v (Fig. 7.9). Determine the amplitude of vertical vibrations of a body joined elastically with the axle at the velocity: a) v = 5 m/s, b) v = 20 m/s.

The parameters are: mass of the body m = 100 kg, spring stiffness k = 19620 N/m (it corresponds to static deformation of the spring by the mass: $y_{stat} = mg/k$ = 100×9.81/19620 = 0.050 m). The waviness of the road can be approximated by sinus function with amplitude a = 20 mm and the pitch of waves $2L$ = 2,0 m.

Solution. Circular frequency of free oscillation $\omega = \sqrt{(k/m)} = \sqrt{(19620/100)}$ = 14.01 s^{-1}; the corresponding natural frequency of movement is $f = \omega/(2\pi) = 14.01/(2\pi)$ =

2.229 Hz. Circular frequency of forced oscillation in the case (a) with velocity v = 5.0 m/s is $\Omega = \pi\, v/L = \pi \times 5.0/1.0 = 15.708\ \mathrm{s}^{-1}$. The corresponding frequency of vertical component of the movement is f_v = 2.5 Hz. Inserting these values into Eq. (7.44) gives

$$y_0 = \left| a \frac{1}{1-(\Omega/\omega)^2} \right| = \left| 0.02 \frac{1}{1-(15.71/14.01)^2} \right| = 0.078 \text{ m} = 78 \text{ mm.}$$

In the case (b), with v = 20.0 m/s, the circular frequency of excitation will be $\Omega = 62.83\ \mathrm{s}^{-1}$, and amplitude of vertical vibration drops to y_0 = 0,001 m = 1 mm.

The exciting circular frequency Ω in both cases was higher than the natural circular frequency of free vibrations ω, so that the calculated values were negative. During slower ride the exciting frequency was near to the natural frequency, and the vibration amplitude was nearly four times larger than the amplitude of road waves. At higher velocity, the exciting frequency was four-and-half times higher than the natural one, and the amplitude of vertical oscillation dropped to a fraction of road waves amplitude.

7.6 Transverse vibration of beams

The situation can be illustrated on a thin elastic beam with an attached body, whose mass m is much higher than the mass of the beam (Fig. 7.10). The situation is thus similar to the material point on an immaterial spring according to Figure 7.1, and similarly, also Equation (7.3) holds, where

$$k = F / y \tag{7.45}$$

is now bending stiffness of the beam, which corresponds to the force pertaining

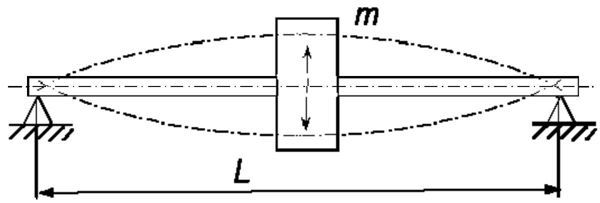

Figure 7.10. *Transverse vibration of a beam with a mass.*

to the deflection of unit magnitude. For example, the deflection of a beam with the load in the middle of the span is [8]:

$$y = Fl^3/(48EJ)\ ; \qquad (7.46)$$

l is the beam length, E is modulus of elasticity, and J is the moment of inertia of the cross section. Comparison of equations (7.45) and (7.46) gives

$$k = 48EJ/l^3\ , \qquad (7.47)$$

so that the determination of the natural frequency is easy. Circular frequency of free vibrations ω is given by Equation (7.3). If harmonic force

$$F(t) = F_0 \sin(\Omega t) \qquad (7.48)$$

acts permanently at the beam centre (Ω is its circular frequency), the beam, after attenuation of transient phenomena, will oscillate with circular frequency Ω and amplitude

$$y_0 = F_0\, l^3/(48EJ)\ . \qquad (7.49)$$

Angular velocity Ω is related to the frequency f of forced vibration as follows:

$$\Omega = 2\pi f\ . \qquad (7.50)$$

The relationship of deflection and the excitation frequency can be seen in Equation (7.18) and Figure 7.5. The force transmission to the supports is given by Eq. (7.37) and Figure 7.8.

Real beams have always some mass, which influences the natural frequency of vibrations of the system „beam + load“, as it was shown in the remark in Chapter 7.2. As an example, a pedestrian bridge across the Thames in London (Millenium Bridge) can be named, which was brought to vibrations by pedestrian walking [14]. The problem became obvious after the footbridge was put into operation. The installation of additional dampers increased the total costs by 30%.

The problems of beam vibration in complex cases, as well as vibration of other bodies go beyond the scope of this book, and the reader is adviced to special literature, e.g. [1 – 3]. Demanding technical problems (natural frequencies and normal modes of vibration of various bodies) are usually solved by the finite element method and the relevant software, mentioned at the end of Chapter 7.12.

7.7 Circular vibrations

Let us investigate a rotating immaterial shaft with a disc of mass m in the middle of the span. The centre of gravity lies exactly on the axis connecting both bearings. If we shortly deflect the shaft (Fig. 7.11), centrifugal force F_o will act on the mass,

$$F_o = my\Omega^2 \; ; \tag{7.51}$$

y is the deflection, and Ω is angular velocity of rotation. This force is held by the bearings. If the rotor is in the middle of the span, each bearing transmits the force $F_o/2$. The deflection of the shaft is resisted by its bending stiffness k, expressed by Equation (7.47). Transverse force, needed for causing the deflection y, is

$$F_r = y\,k \; . \tag{7.52}$$

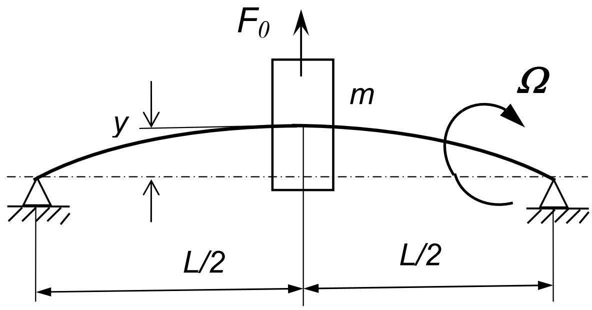

Figure 7.11. *Circular vibration*

This force increases in direct proportion with the deflection y, while the centrifugal force grows with the square of angular velocity. At low velocities, the force for causing deflection y is higher than the centrifugal force, so that after passing the impulse that caused the deflection, the shaft becomes straight again. At some velocity, the centrifugal force is in equilibrium with the reaction force, the deflection is retained, and the shaft with the disc m rotates along a circle of radius y_c. The corresponding angular velocity Ω_c, obtained from the equation of force equilibrium, $F_o = F_{r,c}$, is

$$\Omega_c = \sqrt{k/m} \; . \tag{7.53}$$

This velocity is the same as the angular velocity ω of free transverse vibration of a shaft with the load m. The deflected beam with the disc moves around the axis connecting both bearings. A circular vibration has arisen, which is stable at this velocity. If the revolutions increase for some reason, the centrifugal force will be higher than the reaction force, the deflection starts increasing (and so also the centrifugal force), and this can lead to an accident.

7.8 Rotation of a shaft with eccentric load

The situation is depicted in Figure 7.12. At rest (Fig. A), the centroid of the body is at the distance e from the line connecting both bearings. If the shaft starts rotating, the body generates the centrifugal force, which causes the shaft deflection by δ. The centroid is at a distance $y = e + \delta$ from the axis, so that the centrifugal force is

$$F_o = m(e + \delta)\Omega^2 . \quad (7.54)$$

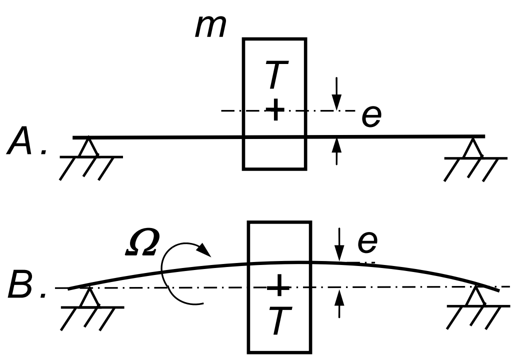

Figure 7.12. *Circular vibration of a beam with eccentric load.*
A – at rest, B – at high velocities above the resonance. T - centre of gravity

This force must have the same magnitude as the force needed to cause the deflection δ, that is $F_r = k\delta$. From their equality,

$$m\,e\Omega^2 = \delta(k - m\Omega^2) , \quad (7.55)$$

the following expression for the deflection follows:

$$\delta = e\,\frac{\Omega^2}{\frac{k}{m} - \Omega^2} = e\,\frac{1}{\frac{\omega^2}{\Omega^2} - 1} \quad (7.56)$$

It is obvious that rotation causes no deflection for zero eccentricity (e = 0). Now, we shall see how the deflection (for $e \neq 0$) varies with the velocity of shaft rotation. If the angular frequency Ω of rotation is lower than the natural frequency ω of transverse vibration of a shaft with a load, the deflection increases with increasing velocity, and for $\Omega \rightarrow \omega$ it would grow above all limits. However, as soon as the velocity of rotation gets above this dangerous region, $\Omega > \omega$, the sign in Equation (7.56) is changed, and the deflection gets smaller with increasing velocity. For very high revolutions (Fig. 12B) the deflection approaches to $-e$. The centre of gravity of the system is now nearer to the axis, connecting both bearings, than at rest. This is so-called self-centering of a shaft with a load.

7.9 Balancing of rotating objects

Different situation pertains to the case with a relatively stiff shaft. Electromotors, turbines, wheels of transport means and numerous machines for machining or processing. Even in home washing machines additional forces arise that load the bearings. The main source is the centrifugal force that arises if the centre of gravity of the rotating body does not lie exactly at the axis of rotation. This was discussed in the preceding section. Moreover, if the rotating mass is not distributed uniformly along this axis, higher reaction forces can appear due to additonal moments. If unfavourable forces should not appear, two conditions must be fulfilled:

1. Centre of gravity must lie on the axis of rotation,
2. Principal axis of inertia must coincide with the axis of rotation.

Also it can be said that the static mass moments to the axis of rotation must equal zero, as well as the product moments of inertia to two planes formed by the axis of rotation and perpendicular one.

These conditions are fulfilled if the body is balanced: 1) statically, and 2) dynamically. In fact, it is impossible to manufacture a perfectly balanced rotational body. Therefore, the bodies that should rotate by high velocity, must be balanced. (Note that dynamic effects increase with the square of angular velocity.) In balancing, a small mass is added (e.g. by welding or screwing) to the body which should be balanced, or, on the contrary, it is taken away (by grinding or drilling). **Static balancing** is easier. It can be done, for example, by laying the rotation body on two parallel horizontal bars („rules“) and observing its behaviour. Statically balanced body stops at any position. If it is not balanced, it starts rolling and stops if its centroid is exactly below the rotation axis. Balancing is achieved by gradual removing material here, or adding some mass on the opposite side. However, even this is not always sufficient for avoiding additional forces in operation. For their complete elimination the body must also be balanced dynamically, so that its principal axis of inertia coincides with the axis of rotation. The purpose of **dynamic balancing** is the removal of moments caused by any unbalance. In this case material is added or taken away in two distant planes perpendicular to the axis of rotation so that the vibrations during the rotation of the body on elastic mounting are minimal. Dynamic balancing needs that the body rotates. If it is balanced dynamically, it is also balanced statically.

Various equipment is used for balancing. The apparatus measures amplitudes and phases of vibration of a rotating body, or dynamic reactions. A suitable computer program evaluates the measurement and shows the unbalance ($m \times r$) corresponding to the centrifugal force at unit angular velocity, or it shows the necessary size of the additional mass and place for its fixing.

A special way of balancing is used in spinning of modern home washing machines. They have acceleration sensor built in the drum bearing. The filled drum is first set turning at lower speed. If the measured unbalance is too high, the machine tries, using very slow rotation forward or back, to achieve better distribution of the washed things (without any external help). Usually one or several attempts are successful, and then the drum is accelerated to the full spinning velocity.

More about balancing can be found, for example, in [9]. Information on some balancing machines are available on web pages of manufacturers, e.g. [10 – 13].

7.10 Energy in vibrating systems with damping

If damping is present, achieved by friction or by flow of a viscous liquid, the supplemented energy is dissipated and changed into heat. In contrast to purely elastic vibration, where the response is practically instantaneous, so that the force and displacement are in phase, the displacement y during vibration with damping is delayed behind the force F (Figure 7.13). This is called hysteresis.

We can assume sinus course of force F and displacement:

$$F(t) = F_0 \sin(\omega t) , \qquad (7.57)$$

$$x(t) = x_0 \sin(\omega t - \varphi) . \qquad (7.58)$$

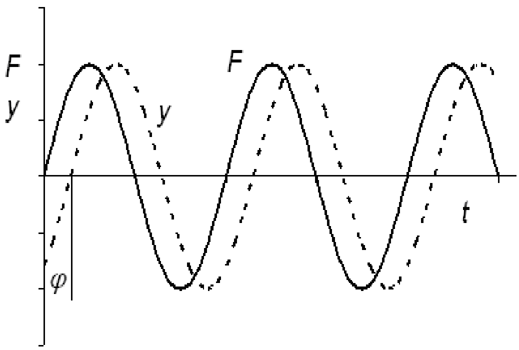

Figure 7.13. *Vibration with damping (hysteresis).*

The differential of work is

$$dW = F\,dx = F_0 \sin(\omega t)\, x_0\, \omega \cos(\omega t - \varphi)\, dt \ , \tag{7.59}$$

and the work done during one period $T = 2\pi/\omega$ is

$$W = \int F_0\, x_0\, \omega \sin(\omega t) \cos(\omega t - \varphi)\, dt = \pi F_0\, x_0 \sin \varphi \tag{7.60}$$

This work changes into heat. If the path is in phase with the force (purely elastic vibration, i.e. $\varphi = 0$), $\sin \varphi = 0$ and no work is dissipated during one cycle.

A good idea on the energy consumption in a loading cycle is obtained from the diagram in coordinate system „force – path“. In our case it is an ellipse with inclined axes (Fig. 7.14). The work done in one cycle is proportional to the area of this ellipse. If the path is in phase with the force ($\varphi = 0$), the ellipse degenerates to an oblique line, and the work done in one cycle equals zero. The most energy is dissipated if the path is delayed behind the force by 90°, as $\sin \varphi = 1$ in Eq. (7.60).

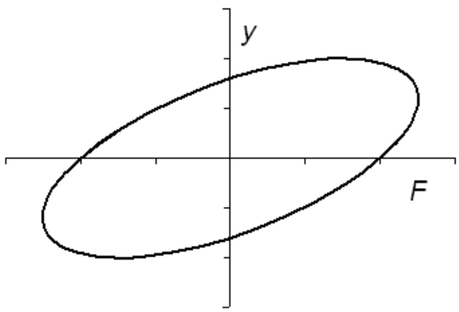

Figure 7.14. *Hysteresis loop.*

7.11 Dynamic absorber of vibrations

Until now we investigated the situations where harmonic force acts on a resiliently mounted body. We have seen that if the frequency of the exciting force approaches to the resonant frequency of the system, the motion amplitude grows, and this could cause damage or destruction of the appliance. We have also shown three ways how to avoid this: 1) change of the exciting frequency (this is not always possible), 2) change of the stiffness of the mounting so that the natural frequency of the system is sufficiently far from the exciting frequency, or 3) addition of an additional damper, for example friction or hydraulic one. Here, the fourth solution using a **dynamic absorber of vibrations** will be shown. The basic idea is that a further body is attached elastically to the body, whose vibrations should be reduced. The exciting force acts on the first body. With suitable tuning only the second, attached body will vibrate, while the first one remains at rest. The theoretic explanation of this behaviour follows [1 – 3].

The situation is depicted in Figure 7.15. The body, whose vibration should be eliminated, has subscript 1, and the dynamic absorber has subscript 2. This system has two degrees of freedom. For simplicity, we shall investigate a system without damping, formed only by inertial masses and forces of springs. The motion equations for both bodies are

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_0 \sin(\Omega t + \varphi_{p1}) \tag{7.61}$$

$$m_2 \ddot{x}_2 + k_2(x_2 - x_1) = 0 \tag{7.62}$$

This is a system of two differential equations. Equation (7.62), with zero on right side, is homogeneous, while Equation (7.61), with nonzero right side, is nonhomogeneous. The general solution is the sum of the solution of homogeneous equation and particular solution of the equation with right side. Here, we shall look

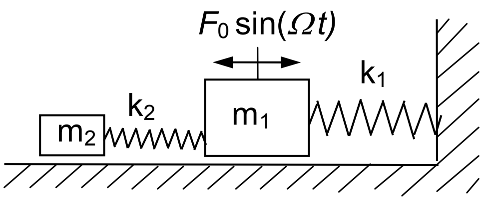

Figure 7.15. *Dynamic absorber of vibrations.*

at the particular solution, as only it corresponds to the steady state. This solution can be assumed in the form

$$x_1 = A_1 \sin(\Omega t + \varphi_{p1}) , \quad x_2 = A_2 \sin(\Omega t + \varphi_{p1}) . \qquad (7.63)$$

After expressing x_1 and x_2 in equations (7.61) a (7.62) by means of expressions (7.63) and their derivatives, and after dividing the expressions by $\sin(\Omega t + \varphi_{p1})$ and considering that the second derivative of sinus is minus sinus, we get the following linear equations:

$$-A_1\Omega^2 m_1 + k_1A_1 + k_2(A_1 - A_2) - F_0 = 0 \qquad (7.64)$$

$$-A_2\Omega^2 m_2 + k_2(A_2 - A_1) = 0 \qquad (7.65)$$

The solution of this system is

$$A_1 = \frac{F_0\,(k_2 - m_2\Omega^2)}{(k_1 - m_1\Omega^2)(k_2 - m_2\Omega^2) - k_2 m_2\Omega^2} \qquad (7.66)$$

$$A_2 = \frac{F_0\,k_2}{(k_1 - m_1\Omega^2)(k_2 - m_2\Omega^2) - k_2 m_2\Omega^2} \qquad (7.67)$$

Let us look at the first expression. If the numerator of the fraction (7.66) equals zero, then the amplitude A_1 equals zero. This means that body 1 will not vibrate at all ! This occurs, if

$$k_2 / m_2 = \Omega^2 , \quad \text{resp.} \quad \Omega = \sqrt{(k_2/m_2)} . \qquad (7.68)$$

In this case, the added mass 2 acts on mass 1 by the same force as the external exciting force, but in the opposite direction. The external force F and spring force $k_2(x_1 - x_2)$ therefore eliminate mutually, and the movement x_1 of mass 1 is zero. (However, body 2 vibrates!) Should this be achieved, the resonant frequency of free vibration of body 2 must be the same as the exciting frequency of force F_0. This can be achieved by suitable choice of the mass m_2 of body 2 and stiffness of spring k_2.

However, the situation is more complex. Dynamic absorber works well only at a certain frequency. Vibrations of body 1 are fully eliminated if the circular frequency of excitation equals ω. The situation for other frequencies is worse. As it follows from a detailed analysis [2], two new regions of resonance appear, with pronounced vibrations, though significantly weaker than the vibrations of body 1 without damping. If various exciting frequencies can appear in operation, an additional damping should be used, or the vibration absorber created as tunable.

Dynamic absorbers of vibration are used not only in machines, but also in very small appliances or in very large structures. Two extreme examples can be named. The first one is vibrating hand shaver [4]. The other is the skyscraper Taipei 101 in Taiwan; the vibration absorber is created as a steel ball of diameter over 5 m and mass 660 t (!), hanged on four thick steel ropes inside the skyscraper on the 88th floor [15]. This arrangement acts as a pendulum, which that has its own frequency.

7.12 Vibration of systems with several degrees of freedom

With the exception of dynamic absorbers, we have dealt with the behaviour of one body on a spring. Its position was described by one coordinate, and its motion in time by one differential equation. In such case one speaks of vibration with one degree of freedom. If two elastic bodies are somehow connected, their vibration is more complex. We have seen it at the dynamic absorber (Fig. 7.15), whose vibrations were described by two differential equations, (7.61) and (7.62); two resonance regions existed here and two natural (own) frequencies of free vibration. This was the case with two degrees of freedom. Here, the solution according to [2] will be shown, and then it will be generalised for more degrees of freedom.

Euqations (7.61) and (7.62) have the folloving form for free vibration:

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \tag{7.69}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \tag{7.70}$$

Now, we can introduce auxiliar constants a, b, c in the following way:

$$a = (k_1+k_2)/m_1\ ,\ \ b = k_2/m_1\ ,\ \ c = k_2/m_2\ , \tag{7.71}$$

Equations (7.69) and (7.70) change with these constants to the following form

$$\ddot{x}_1 + ax_1 - bx_2 = 0 \tag{7.72}$$

$$\ddot{x}_2 - cx_1 + cx_2 = 0 \tag{7.73}$$

If we assume their solutions in the form

$$x_1 = A_1 \sin(\omega t + \alpha)\,, \qquad x_2 = A_2 \sin(\omega t + \alpha)\,, \tag{7.74}$$

and insert them with their second time-derivatives into (7.72) a (7.73), we obtain after a rearrangement

$$A_1(a - \omega^2) \quad - A_2 b \qquad = 0 \tag{7.75}$$

$$-A_1 c \quad + \qquad A_2(c - \omega^2) \; = 0 \tag{7.76}$$

This is a system of two equations with zero right side. The solution with non-zero amplitudes A_1, A_2 exists only if the determinant of the system equals zero, that is if

$$(a - \omega^2)(\,c - \omega^2) - bc = 0\,, \tag{7.77}$$

or

$$\omega^4 - (a + c)\omega^2 + c(a - b) = 0\,. \tag{7.78}$$

This is equation of the fourth degree, but it can also be understood as a quadratic equation of the unknown ω^2. The roots of this equation are two:

$$\omega_{1,2}{}^2 = \frac{a+c}{2} \mp \sqrt{\left(\frac{a+c}{2}\right)^2 - c(a-b)} \tag{7.79}$$

The system with two degrees of freedom has two own circular frequencies, ω_1, ω_2. Similarly, cases with three and more degrees of freedom can exist, for example in vibration of beams, shafts and many other bodies, so that more natural frequencies can exist. (Generally: the number of eigenfrequencies is identical with the number of degrees of freedom of the system.) If cases with more degrees of freedom should be solved, writing of the individual equations according to the system (7.64) and (7.65) would be too complicated, and **matrix algebra** is used instead. A matrix is a set of numbers, arranged in certain way into rows and columns. Similar operations can be made with matrices as with numbers, for example addition or multiplication using formal rules of matrix algebra. The advantage is very simple and illustrative writing. For example, matrix form of Equations (7.61) and (7.62) is as follows:

$$\boldsymbol{m}\,\ddot{\boldsymbol{y}} + \boldsymbol{k}\,\boldsymbol{y} = \boldsymbol{F}(t)\;. \tag{7.80}$$

This is similar to Equation (7.12) for one-dimensional case with a material point on a spring, without damping. The only difference is that now $\boldsymbol{m}$ represents the mass matrix, $\boldsymbol{y}$ is the matrix of displacements, and $\ddot{\boldsymbol{y}}$ is the matrix of their second derivatives, $\boldsymbol{k}$ is the stiffness matrix, and $\boldsymbol{F}$ is the matrix of exciting forces. If damping is present, Equation (7.80) will contain also the matrix of damping $\boldsymbol{b}$ and matrix of velocities $\boldsymbol{y}'$, similarly to Equation (7.12). The arrangement of the individual values in the matrices, and work with them, is a matter of particular computer program. The solution of vibrations with more degrees of freedom needs a computer and suitable software. Also the solution of equations in matrix form is a matter of software, and a common user does not need to know all details.

If a body vibrates in diferent resonant frequencies, its characteristic shapes are different, and one speaks of the **own** (natural) **shapes**. Example of the first three own shapes of a beam on two supports is given in Figure 7.16.

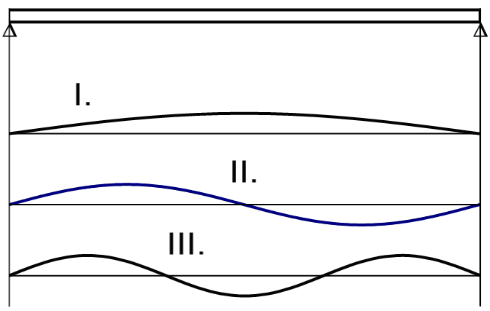

Figure 7.16. *Beam on two supports, and three first own shapes*

Analysis of deformations or vibrations of bodies of complex shape is made today by the **finite element method**, FEM [16, 17]. With this method, the investigated body is divided into many simpler bodies, called finite elements, which are connected at nodes. An element can have form of a bar, beam, a small simple plate, shell, tetrahedron, or a brick, and they all together form a mesh. The unknown quantities are the displacements of the individual nodes. The components of displacements, forces and stresses at the individual points are expressed by means of matrices. The physical principle if solution is search of such values of node displacement, which correspond to the minimum total energy of the system, consisting of the strain energy of the body and the potential energy of all loads. In the final form, the problem is converted to the solution of very large systems of equations (from hundreds to millions). For this purpose, very powerful commercial FEM programs have been created for the analysis of various structures, including the determination of stresses and deformations, as well as solution of dynamic problems and finding own frequencies and shapes. For more, see [18 – 21].

Information on commercial shock absorbers can be found in leaflets and brochures of manufacturers, such as [22, 23].

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8. Dimensional Analysis and Theory of Similarity

In design of structures, machines and various appliances, including those for impact mitigation, dimensional analysis and theory of similarity are very useful, as they simplify experiments, spare experimental work and make the results more general. (This also holds for experiments performed via computers.) This chapter, based on works [1 – 4], shows various kinds of similarity and gives examples of non-dimensional quantities and instructions for their creation.

8.1 Dimensional analysis

Every physical quantity is described by a numerical value accompanied by a unit. The numerical value says how many times the considered quantity is larger than its unit. An example of length is 5.3 m, example of force is 25 N, of time is 15.6 ms. In addition to the fundamental units (meter, kilogram, second…), defined in the Système International (SI), also various derived units are used, as well as prefixes (μ, m, k, M…), denoting the order.

Every equation, describing a physical phenomenon, must be dimensionally homogeneous: its left side must have the same dimension as the right side. The **check of homogeneity** should always be done before the first use of a newly derived formula. Such check also helps in formulating a correct relationship among the variables. Consider, for example, a formula for the deflection y of an elastic beam loaded by a force F. It is known from mechanics of materials that y will be directly proportional to F and indirectly proportional to the bending stiffness of the beam, defined as $E \times J$, where E is the Young modulus of the material and J is the moment of inertia of the cross section. The deflection will also be proportional to some power S of the beam length L. Now, imagine that we do not know the exponent S. In such case we can write the basic form of the formula:

$$y = C \times F \times L^S / (E \times J) \; ; \qquad (8.1)$$

C is a non-dimensional constant. Replacement of the individual quantities in Eq. (8.1) by their units gives

$$\mathrm{m} = 1 \times \mathrm{N} \times \mathrm{m}^{S}/(\mathrm{Nm}^{-2} \times \mathrm{m}^{4})$$

The dimension of the right side must be the same as that of the left side, i.e. meter, or, generally, m^1. The product of all terms containing m is $\mathrm{m}^{S}\times\mathrm{m}^{2}\times\mathrm{m}^{-4} = \mathrm{m}^{S+2-4} = \mathrm{m}^{S-2}$. Comparison of the exponents on the left and right side of the equation gives $1 = S - 2$. From this it follows $S = 3$, so that $y = C\times F\times L^3/(EJ)$, a formula well known from mechanics.

If one side of an equation is created by a sum of several terms, they all must have the same dimension. For example, vertical movement y of a body falling in gravitational field is described as

$$y = y_0 + v_0 t + \tfrac{1}{2} g t^2 \qquad (8.2)$$

t is time, y_0 and v_0 are the position and velocity of the body at $t = 0$, and g is the acceleration of gravity. The dimensional homogeneity demands that the individual quantities cannot exist in the physical equation independently, but only in groups of the same dimension. If Equation (8.2) is divided by one of the terms, for example y_0, it changes to non-dimensional form

$$y/y_0 = 1 + v_0 t/y_0 + \tfrac{1}{2} g t^2/y_0 \qquad (8.3)$$

with normalised quantities y/y_0, $v_0 t/y_0$ and $g t^2/y_0$.

Nearly every physical equation can be transformed to non-dimensional form. The use of normalised quantities has many advantages. Physical equations, expressed by means of non-dimensional variables, are more general than if they are expressed by dimensional quantities. The relative displacement, y/y_0, does not depend simply on v_0, t and y_0, but only on their certain combinations, shown in Eq. (8.3). Dimensionless quantities thus enable one to combine the results of experiments made with specimens of various initial velocity and position, the only condition being their proper combination. (In the above case of a beam, combination of its size and material play a role.) Therefore, more data and a wider range of parameters can be used for the formulation of a certain law. The results expressed in non-dimensional form are also more universal, valid for the whole class of similar objects, with similar geometry or physical properties. Moreover – and this is very important – the use of non-dimensional quantities can spare experimental work, because

The relationship of *N* quantities, whose dimensions can be expressed by means of *D* basic dimensions, may usually **be replaced by a relationship of only**

$$\boldsymbol{P = N - D} \tag{8.4}$$

dimensionless parameters *Π*.

According to this ***Buckingham theorem***, the determination of fewer regression constants needs fewer experiments. The reduction of experimental work is significant especially if the investigated relationship contains many quantities and if the number of variables, N, is close to the number of basic dimensions, D. This can be illustrated on the previous example of a falling body. Equation (8.2) represents a relationship of 5 quantities: y, y_0, v_0, g and t; that is $N = 5$. These quantities can be expressed by means of two basic dimensions: meter and second; thus $D = 2$. According to Eq. (8.4), the number of non-dimensional parameters should be $P = N - D = 5 - 2 = 3$. And really, Equation (8.3) is the relationship of 3 dimensionless parameters: y/y_0, v_0t/y_0 and gt^2/y_0. The advantage of non-dimensional form will be more obvious from the next example. If the influence of six factors should be investigated, each on two levels (low and high), the number of necessary experiments would be $2^6 = 64$. If the number of dimensionless factors were only 4, the number of necessary experiments drops to $2^4 = 16$, i.e. to 25%!

8.2 Similarity

The use of non-dimensional quantities is also of prime importance in the study of behaviour of real objects by means of models. For example, building of a new large ship or a bridge is accompanied by many uncertainties, and the potential losses due to wrong design would be very high. Therefore, usually a smaller model is built first and tested. However, if the model should adequately reflect the behaviour of the actual structure, similarity between them must exist. There are various kinds of similarity, for example:

Geometric similarity, which means identity of shape, equality of corresponding angles, and a constant proportionality between the corresponding dimensions (so-called scale factor). The following relation holds:

Model dimension = Scale factor* × *Dimension of the real object

For example, a model of a building, made in the scale 1:20, has all dimensions 20-times smaller than the real building.

Static similarity means that the relative deformations of a model under constant stress are in the same proportion as the corresponding deformations of the object.

Kinematic similarity is based on the ratio of the time proportionality between corresponding events in the model and the object.

Dynamic similarity exists if the forces acting at corresponding times and locations in the model and object are in a fixed ratio.

The **theory of similarity** works with so-called **similarity numbers**. Those, who have attended a college course of physics, know, for example, the Reynolds number (Re), which helps in assessing whether a flow of a liquid is laminar or turbulent. More examples are given at the end of this chapter. The similarity numbers are dimensionless. In fact, every non-dimensional quantity can serve as a similarity number.

Dimensionless variables can be created in various ways. The simplest case is the ratio of some quantity to its characteristic value, for example x/x_0 or $\Delta x/x_0$ for distance or displacement. Well known in mechanics are: strain, defined as relative elongation ($\varepsilon = \Delta L/L$), Poisson number μ (the ratio of relative shortening in transverse direction to the relative elongation in the direction of stress action), or coefficient of friction f, defined as the ratio of the force, needed to slide a body along another body, and the normal force pressing both bodies together. Another example is the relative position of a point in a body, for example

$$\xi = (x - x_{min})/(x_{max} - x_{min}) \qquad (8.5)$$

x, x_{max} and x_{min} represent the coordinates. Similarly it is possible to express time. Non-dimensional temperature, $\theta = (T - T_\infty)/(T_0 - T_\infty)$, is used for universal description of processes of heat transfer (T_0 is the initial temperature and T_∞ is the final temperature). In this case also the position of the investigated place and the time can be in non-dimensional form. Formal procedures can be found in [1 – 4].

Dimensionless must also be the arguments in mathematical functions of type sin, cos, log, exp. Otherwise any change of the units would change the numerical value of the result. Non-dimensional are also the arguments in probability distributions. For example, normal distribution uses the argument $\{½[(x - \mu)/\sigma]^2\}$, where μ and

σ are the mean value and standard deviation, respectively. However, the term in square brackets is nothing else than standardised variable, which expresses the distance of x from the mean value μ as the multiple of standard deviation σ.

NOTE: Non–dimensional quantities are used more often than we realise!

8.3 Further recommendations

1) Sometimes the form of a non-dimensional parameter does not correspond to our intentions or experimental possibilities. Generally, it is possible to create new parameters (or similarity numbers) by making a product or ratio of the original ones, or to change them by making their reciprocal or some power. As they are dimensionless, the new parameters obtained by such transformations will be dimensionless, too.

2) If several quantities of the same dimension appear in one problem, it is also possible to create non-dimensional parameters directly as their ratios. This can reduce the number of arguments. This will be illustrated on an example of the deflection y of a beam with rectangular cross section ($w \times h$) and length L loaded by a point force F. The modulus of elasticity is E. The variables and their dimensions are: y(m), w(m), h(m), L(m), F(N), E(Nm^{-2}); that is 6 variables with 2 dimensions. The number of non-dimensional parameters needed for the description of the problem is $F = N - D = 6 - 2 = 4$. We can immediately create three parameters $\Pi_1 = y/h$, $\Pi_2 = b/h$ and $\Pi_3 = L/h$. Two quantities remain (F and E), which must be contained in the fourth parameter. With respect to their dimensions and the condition of non-dimensionality also one geometric quantity must be included in Π_4, for example h or its power. We obtain this parameter as $\Pi_4 = F/(Eh^2)$. The studied relationship can thus be written in the following non-dimensional form:

$$y/h = f\,[F/(Eh^2), L/h, w/h] \qquad (8.6)$$

One should remember that not the individual quantities L or F, etc. are important for the study of relative deflection y/h, but their ratios.

3) In some problems always non-dimensional quantities appear. Examples are coefficient of friction, Poisson number μ for lateral contraction, or angle φ (rad). These quantities automatically become arguments in the dimensionless relationships.

4) When creating dimensionless parameters, one can use the existing knowledge on the investigated or similar problem. For example, we may know that the deflection of an elastic beam is directly proportional to the load and indirectly to the modulus of elasticity. Sometimes, analytical solution is known for very small or very large values of certain variable. This can help in searching for a proper form of the arguments. Sometimes it is known that some quantities must appear in certain combination. This combination can be considered as a new variable, which can enable reduction of the total number of variables. Consider, for example, force acting in the contact area of two bodies. If friction should be investigated, the force F (N) and contact area A (m^2) can be replaced by contact pressure $p = F/A$ (N/m^2).

5) When an experiment should be prepared, it is necessary to include all quantities, which can play a role. Otherwise wrong and misleading results can be obtained. It is less dangerous to include a quantity, whose importance is uncertain (and, perhaps, it appears later that it may be omitted), than to omit a quantity, which could later be found as important. The use of dimensional analysis sometimes reveals serious shortcomings. For example, if some dimension appears only at one quantity, this quantity falls out and will not be included in any non-dimensional parameter. However, if this quantity is obviously necessary for the description of the investigated phenomenon, it is necessary to add another quantity having the same dimension. This can be illustrated on a study of wear rate of a brake pad. The quantities playing a role are: wear rate w (m/s), velocity of mutual sliding v (m/s), and the pressure in the contact area p (N/m^2). The non-dimensional parameter can be searched in general form

$$\Pi = w^{x1}\, x^{x2}\, p^{x3}. \tag{8.7}$$

This expression can be rewritten by means of the dimensions of the individual quantities (m, s, N):

$$[\mathrm{m}]^0\,[\mathrm{s}]^0\,[\mathrm{N}]^0 = [\mathrm{m}\times\mathrm{s}^{-1}]^{x1} \times [\mathrm{m}\times\mathrm{s}^{-1}]^{x2} \times [\mathrm{N}\times\mathrm{m}^{-2}]^{x3} \tag{8.8}$$

The left side corresponds to non-dimensional notation. It follows from the condition of equality of exponents at the same base, $N^0 = N^{x3}$, that $x_3 = 0$. But it is well known from experiments that the wear rate does depend on the contact pressure p, so that x_3 cannot equal 0. It is thus necessary to include one further quantity, which would also have the dimension Nm^{-2}. This could be, for example, hardness H (Nm^{-2}), which characterises the resistance of the material. Now, the general form of the non-dimensional parameter is

$$\Pi = w^{x1}\, v^{x2}\, p^{x3}\, H^{x4} \tag{8.9}$$

From this expression, we can easily formulate the appropriate relationship of dimensionless parameters as ***w/v = f(p/H)***, and perform a series of experiments in order to find the appropriate form of the function *f*.

8.4 Limitations of similarity principle

The principle of similarity holds only under some conditions, and outside them it loses its validity. A good example is the transition from elastic to elastic-plastic deformations in components from ductile materials. If the stresses are lower than the yield strength, the deformations are elastic; linear relationship exists between stresses and strains, and the deformations and stresses caused by several loads can be calculated as the sum of deformations or stresses caused by the individual loads. However, the laws for elastic-plastic deforming are nonlinear and the situation in the particular case must be solved for all loads acting simultaneously. Another case is the strength dependence of brittle components on the size of loaded area or volume. Brittle fracture usually starts at a pre-existing weak point. Smaller size of the loaded area or volume means a lower probability of occurrence of a larger defect. A smaller defect can act as a starting point only at higher stress level. Therefore, very small objects are stronger. For similar reasons, also the fatigue limit of metal components increases with their decreasing size.

Sometimes, the studied problem can simultaneously contain quantities that depend on different powers of another quantity. For example, the energy dissipated during fracture, is proportional to the fracture area (m^2), while strain energy, accumulated in the body, is proportional to its volume (m^3). If mitigation of impacts should be investigated on a model of different size, we must consider what is the main mechanism of the energy dissipation, consider whether both components are equally important, and – if possible – to neglect one of them.

The processes during fast plastic deforming sometimes depend on strain rate. If we want to study the effect of impact load on a model of smaller dimensions (L_m) than the actual object (L_p), we must not forget that if the same strain rate should exist in both cases, different velocities of impact should be used, so that it holds

$$v_{0m}/v_{0p} = L_m/L_p \;; \tag{8.10}$$

the subscripts m and p denote the model and prototype.
One must also have in mind that sometimes the investigated quantity changes with the changes of a certain parameter relatively slowly, but from its certain level it can change very quickly. The relationship, describing some behaviour or process, is often valid only within certain range of parameters. If the pertinent process is described by means of non-dimensional quantities, the conditions for a transition from one mode to another are characterised by a **critical value** of some of these quantities. A well-known example is the change from laminar to turbulent flow at the critical value of Reynolds number. One must therefore always consider all possible influences, and reduce their number only after a thorough analysis.

8.5 Examples of dimensionless quantities

Material properties

E_1/E_2, H_1/H_2	ratio of elastic moduli or hardness; subscripts denote the components
$E(x)/E_0$, $H(x)/H_0$	ratios as above, subscript 0 denotes the characteristic value
H/Y, E/Y, E/H	ratio of hardness and yield strength or elastic modulus
σ/Y, σ/σ_u, Y/σ_u	ratio of stress to yield strength Y or ultimate strength σ_u

Geometry

x/d	x – distance, deformation
$\Delta l/L$	relative displacement or elongation, L – basic length

Forces and stresses

F/F_0	ratio of load F and characteristic force
σ/σ_m	ratio of the stress σ to the mean stress or pressure σ_m

Time

t/t_0	t_0 – characteristic time (t. of load increase, stopping, etc.).

8.6 Examples of similarity numbers

Important similarity numbers were given names of prominent scientists, and are denoted by the first two letters of the pertinent name. Some examples follow.

Euler	$\mathrm{Eu} = \Delta p / ru^2$; Δp – pressure difference, ρ – density, u – characteristic velocity
Fourier	$\mathrm{Fo} = a\tau/d^2$; a – thermal diffusivity, τ – time, d – characteristic dimension
Froude	$\mathrm{Fr} = u^2/gd$; u – characteristic velocity, g – acceleration of gravity, d – characteristic dimension
Galilei	$\mathrm{Ga} = gd^3/\nu^2$; g – acceleration of gravity, d – characteristic dimension, ν – kinematic viscosity
Nusselt	$\mathrm{Nu} = \alpha d/\lambda$; α – coefficient of heat transfer, d – characteristic dimension, λ – coefficient of thermal conductivity of the surrounding medium
Péclet	$\mathrm{Pe} = ud/a$; u – velocity, d – characteristic dimension, a – thermal diffusivity
Prandtl	$\mathrm{Pr} = \nu/a$; ν – kinematic viscosity, a – thermal diffusivity
Reynolds	$\mathrm{Re} = ud\rho/\eta = ud/\nu$; u – characteristic velocity, d – characteristic dimension, ρ – density of the liquid, η – dynamic viscosity, $\nu = \eta/\rho$ = kinematic viscosity
Stokes	$\mathrm{Stk} = ut/d$; u – velocity, t – relaxation time, d – characteristic dimension

References to Chapter 8

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Index (the numbers in brackets denote the chapters)

Book title:	Impacts and vibrations. Principles of mechanics and mitigation
Author:	prof. Ing. Jaroslav Menčík, CSc.
Publisher:	University of Pardubice
Print:	September 2018
No. of pages:	132
No. of copies:	20
Edition:	first
Printed by:	Polygrafické středisko Univerzity Pardubice

Tato publikace neprošla jazykovou korekturou.

ISBN 978-80-7560-165-0 (print)

ISBN 978-80-7560-166-7 (pdf)

The book is accessible on http://hdl.handle.net/10195/71823 or https://e-shop.upce.cz

ISBN 978-80-7560-165-0 (pdf)