$\mathcal{I}_c^{(q)}$-convergence of arithmetical functions

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*Dedicated to the memory of Professor Tibor Šalát (*1926 – †2005)*

Abstract

Let $n > 1$ be an integer with its canonical representation, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Put $H(n) = \max\{\alpha_1, \dots, \alpha_k\}$, $h(n) = \min\{\alpha_1, \dots, \alpha_k\}$, $\omega(n) = k$, $\Omega(n) = \alpha_1 + \cdots + \alpha_k$, $f(n) = \prod_{d|n} d$ and $f^*(n) = \frac{f(n)}{n}$. Many authors deal with the statistical convergence of these arithmetical functions. For instance the notion of normal order is defined by means of statistical convergence. The statistical convergence is equivalent with $\mathcal{I}_d$-convergence, where $\mathcal{I}_d$ is the ideal of all subsets of positive integers having the asymptotic density zero. In this paper we will study $\mathcal{I}$-convergence of well known arithmetical functions, where $\mathcal{I} = \mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} \, : \, \sum_{a \in A} a^{-q} < +\infty\}$ is an admissible ideal on $\mathbb{N}$ for $q \in (0, 1\rangle$ such that $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_d$.

1 Introduction

The notion of statistical convergence was introduced in [6], [24] and the notion of $\mathcal{I}$-convergence from the paper [15] coresponds to the natural generalization of statistical convergence (see also [4] where $\mathcal{I}$-convergence is defined by means of filter-the dual notion to ideal). These notions have been developed in several directions in [2], [3], [5], [9], [13], [14], [19], [22] and have been used in various parts of mathematics, in particular in number theory and ergodic theory, for example [1], [7], [10], [11], [14], [18], [20], [21], [23]. Recall the definition and some examples of ideals on $\mathbb{N}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$. $\mathcal{I}$ is called an admissible ideal of subsets of positive integers, if $\mathcal{I}$ is additive (if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$), hereditary (if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$), containing all singletons and it does not contain $\mathbb{N}$. Here we present some examples of admissible ideals.

More examples can be found in the papers [11], [13] and [17].

Example 1.1. a) The class of all finite subsets of $\mathbb{N}$ forms an admissible ideal usually denoted by $\mathcal{I}_f$.

b) Let ϱ be a density function on $\mathbb{N}$, the set $\mathcal{I}_\varrho = \{A \subseteq \mathbb{N} \ : \ \varrho(A) = 0\}$ is an admissible ideal. We will use namely the ideals $\mathcal{I}_d$, $\mathcal{I}_\delta$, $\mathcal{I}_u$ and $\mathcal{I}_h$ related to asymptotic, logarithmic, uniform and Alexander density respectively. The definitions for those densities see [1], [8], [11], [13], [17] and [26].

c) For an $q \in (0,1\rangle$ the set $\mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} \ : \ \sum_{a \in A} a^{-q} < +\infty\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \{A \subseteq \mathbb{N} \ : \ \sum_{a \in A} a^{-1} < +\infty\}$ is usually denoted by $\mathcal{I}_c$. It is easy to see, that for any $q_1, q_2 \in (0,1)$, $q_1 < q_2$ we have

$$\mathcal{I}_f \subsetneq \mathcal{I}_c^{(q_1)} \subsetneq \mathcal{I}_c^{(q_2)} \subsetneq \mathcal{I}_c \subsetneq \mathcal{I}_d \subsetneq \mathcal{I}_\delta. \tag{1}$$

d) Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$ be a decomposition on $\mathbb{N}$ (i.e. $D_k \cap D_l = \emptyset$ for $k \neq l$). Assume that D_j ($j = 1, 2, \dots$) are infinite sets (e.g. we can choose $D_j = \{2^{j-1}.(2s-1) \ : \ s \in \mathbb{N}\}$ for $j = 1, 2, \dots$). Denote $\mathcal{I}_\mathbb{N}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of D_j. Then $\mathcal{I}_\mathbb{N}$ is an admissible ideal.

Let us recall notions of $\mathcal{I}$- and $\mathcal{I}^*$-convergence of sequences of real numbers see [15].

Definition 1.2. (i) We say that a sequence $x = (x_n)_{n=1}^{\infty}$ $\mathcal{I}$-converges to a number L and we write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \ : \ |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.

(ii) Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$, if there is a set $H \in \mathcal{I}$, such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots\}$ we have

$$\lim_{k \to \infty} x_{m_k} = L,$$

where the limit is in the usual sense.

It is clear that for an admissible ideal $\mathcal{I}$ we have that $\mathcal{I}^*$-convergence of sequence implies $\mathcal{I}$-convergence. The converse is not true, for example the ideals $\mathcal{I}_u = \{A \subseteq \mathbb{N} \ : \ u(A) = 0\}$, where u is the uniform density (see [8], [26]), $\mathcal{I}_\mathbb{N}$ from example 1.1 d) (see [15]) and the ideal $\mathcal{I}_\mu = \{A \subseteq \mathbb{N} \ : \ \mu(A) = 0\}$, where μ is the Buck's measure (see [17]) have this property. For ideals $\mathcal{I}_d$ and $\mathcal{I}_\delta$ the notions $\mathcal{I}$- and $\mathcal{I}^*$-convergence are equivalent (see [15]). The following theorem shows that also for all ideals $\mathcal{I}_c^{(q)}$ for $q \in (0,1\rangle$ the concepts $\mathcal{I}$- and $\mathcal{I}^*$-convergence coincide.

Theorem 1.3 (Theorem 1.5 from [11])**.** *For any $q \in (0,1\rangle$ the $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$-convergence are equivalent.*

Proof. It suffices to prove that for any sequence $(x_n)_{n=1}^{\infty}$ of real numbers such that $\mathcal{I} - \lim x_n = \xi$ there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus M \in \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = \xi$.

For any positive integer k let $\varepsilon_k = \frac{1}{2^k}$ and $A_k = \{n \in \mathbb{N} : |x_n - \xi| \geq \frac{1}{2^k}\}$. As $\mathcal{I} - \lim x_n = \xi$, we have $A_k \in \mathcal{I}$, i.e.

$$\sum_{a\in A_k} a^{-q} < \infty.$$

Therefore there exists an infinite sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of integers such that for every $k = 1, 2, \ldots$

$$\sum_{\substack{a>n_k \\ a\in A_k}} a^{-q} < \frac{1}{2^k}.$$

Let $H = \bigcup_{k=1}^{\infty}[(n_k, n_{k+1}\rangle \cap A_k]$. Then

$$\sum_{a\in H} a^{-q} \leq \sum_{\substack{a>n_1 \\ a\in A_1}} a^{-q} + \sum_{\substack{a>n_2 \\ a\in A_2}} a^{-q} + \cdots + \sum_{\substack{a>n_k \\ a\in A_k}} a^{-q} + \cdots <$$

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots < +\infty.$$

Thus $H \in \mathcal{I}$. Put $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots < m_k < \cdots\}$. Now it suffices to prove that $\lim_{k\to\infty} x_{m_k} = \xi$. Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < \varepsilon$. Let $m_k > n_{k_0}$. Then m_k belongs to some interval $(n_j, n_{j+1}\rangle$ where $j \geq k_0$ and doesn't belong to A_j $(j \geq k_0)$. Hence m_k belongs to $\mathbb{N} \setminus A_j$, and then $|x_{m_k} - \xi| < \varepsilon$ for every $m_k > n_{k_0}$, thus $\lim_{k\to\infty} x_{m_k} = \xi$. □

In [15] was formulated a necessary and sufficient condition for an admissible ideal $\mathcal{I}$ under which $\mathcal{I}$- and $\mathcal{I}^*$-convergence are equivalent. This condition (AP) is similar to the condition (APO) in [5] and [6].

Definition 1.4 (see also [8])**.** An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that symmetric difference $A_j \Delta B_j$ is finite for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Corollary 1.5 (see [11]). *Ideals $\mathcal{I}_c^{(q)}$ for $q \in (0,1\rangle$ have the property (AP).*

It is easy to prove the following lemma.

Lemma 1.6 (see [15]). *If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ then the statement $\mathcal{I}_1 - \lim x_n = x$ implies $\mathcal{I}_2 - \lim x_n = x$.*

The converse is not true as the following example shows.

Example 1.7. $\mathcal{I}_c^{\left(\frac{1}{2}\right)} \subsetneq \mathcal{I}_c$. Define the sequence $x = (x_n)_{n=1}^{\infty}$ as follows: $x_n = 1$ for $n = k^2$ and $x_n = 0$ otherwise. Then $\mathcal{I}_c - \lim x_n = 0$ but $x = (x_n)_{n=1}^{\infty}$ is not $\mathcal{I}_c^{\left(\frac{1}{2}\right)}$-convergent.

Recall some arithmetical functions, which we will investigate with respect to $\mathcal{I}_c^{(q)}$-convergence for $q \in (0,1\rangle$. Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the canonical representation of the integer $n \in \mathbb{N}$.

1. $\omega(n)$ - the number of distinct prime factors of n ($\omega(n) = k$),
2. $\Omega(n)$ - the number of prime factors of n counted with multiplicities ($\Omega(n) = \alpha_1 + \cdots + \alpha_k$),
3. for $n > 1$ denote

   $$h(n) = \min_{1 \le j \le k} \alpha_j, \quad H(n) = \max_{1 \le j \le k} \alpha_j$$

   and $h(1) = 1$, $H(1) = 1$,
4. $f(n) = \prod_{d|n} d$, $f^*(n) = \frac{1}{n} f(n)$, where $n = 1, 2, \ldots$,
5. $a_p(n)$ is defined as follows: $a_p(1) = 0$ and if $n > 0$, then $a_p(n)$ is a unique integer $j \ge 0$ satisfying $p^j \mid n$, but $p^{j+1} \nmid n$ i. e., $p^{a_p(n)} \| n$.

In the papers [7], [21], [23] and in the book [26] there are studied various convergences of above mentioned arithmetical functions. The following equalities were proved in the paper [23] by using the notion of normal order and some results from [12] and [16].

$$\operatorname{limstat} \frac{\omega(n)}{\log \log n} = \operatorname{limstat} \frac{\Omega(n)}{\log \log n} = 1$$

and

$$\operatorname{limstat} \frac{h(n)}{\log n} = \operatorname{limstat} \frac{H(n)}{\log n} = 0.$$

Recall that the statistical convergence coinsides with $\mathcal{I}_d$-convergence, that is why we can write $\mathcal{I}_d - \lim$ instead of limstat in the previous equalities. Similarly for the functions $f(n)$ and $f^*(n)$. In [21] it is proved the following equality:

$$\mathcal{I}_d - \lim \frac{\log\log f(n)}{\log\log n} = \mathcal{I}_d - \lim \frac{\log\log f^*(n)}{\log\log n} = 1 + \log 2.$$

Let us recall one more result from [20] there was proved that the sequence $\left(\log p \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is $\mathcal{I}_d$ convergent to 0. Moreover the sequence $\left(\log p \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is $\mathcal{I}_c^{(q)}$-convergent to 0 for $q = 1$ and it is not $\mathcal{I}_c^{(q)}$-convergent for all $q \in (0,1)$, this was shown in [7]. In [1] it was proved that this sequence is also $\mathcal{I}_u$-convergent to 0. It is known that $\mathcal{I}_u \subsetneq \mathcal{I}_d$ (see for ex. [2], [3]) but the ideals $\mathcal{I}_c$ an $\mathcal{I}_u$ are not disjoint and moreover $\mathcal{I}_u \not\subseteq \mathcal{I}_c$ and $\mathcal{I}_c \not\subseteq \mathcal{I}_u$. For example the set of all prime numbers belongs to $\mathcal{I}_u$ but not belongs to $\mathcal{I}_c$. On the other hand there exists the set $B = \bigcup_{k=1}^{\infty} B_k$, where $B_k = \{k^3+1, k^3+2, \ldots, k^3+k\}$ which not belongs to $\mathcal{I}_u$ but it belongs to $\mathcal{I}_c$.

Under the fact that $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_d$ for all $q \in (0, 1\rangle$ and Lemma 1.6 it is useful to investigate $\mathcal{I}_c^{(q)}$-convergence of these sequences for $q \in (0, 1\rangle$.

2 Main results

In this section we will investigate the $\mathcal{I}_c^{(q)}$-convergence of special sequences described in the introduction. Under the Lemma 1.6 it is clear that if there exists the $\mathcal{I}_c^{(q)}$-limit of some sequence for any $q \in (0, 1\rangle$ then it is equal to the $\mathcal{I}_d$-limit of the same sequence. There are no other options.

First of all consider the sequences $\left(\frac{h(n)}{\log n}\right)_{n=2}^{\infty}$ and $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$. In [23] it was proved that these sequences are dense on $(0, \frac{1}{\log 2})$ and moreover they both are statistically convergent to zero. The same result we have for $\mathcal{I}_c^{(q)}$-convergence, but only for the sequence $\left(\frac{h(n)}{\log n}\right)_{n=2}^{\infty}$ for all $q \in (0, 1\rangle$.

Theorem 2.1. *We have*

$$\mathcal{I}_c^{(q)} - \lim \frac{h(n)}{\log n} = 0, \ \textit{for all } q \in (0, 1\rangle\,.$$

Proof. Let $k \in \mathbb{N}$ and $k \geq 2$. It is easy to see that the following equality holds

$$1 + \sum_{n\,:\,h(n)\geq k} n^{-q} = \prod_{p\in\mathbb{P}} \left(1 + \frac{1}{p^{kq}} + \frac{1}{p^{(k+1)q}} + \cdots\right) \tag{2}$$

where $\mathbb{P}$ denotes the set of all primes.

The right hand side of the equality (2) equals

$$\prod_{p\in\mathbb{P}}\left(1+\frac{1}{p^{kq}}\cdot\frac{1}{1-\frac{1}{p^q}}\right)=\prod_{p\in\mathbb{P}}\left(1+\frac{1}{p^{(k-1)q}\cdot(p^q-1)}\right).$$

Then for $q>\frac{1}{k}$ the product on the right hand side of the previous equality converges. Thus the series on the left hand side of (2) converges.

Let $\varepsilon>0$. Put $A(\varepsilon)=\left\{n\ :\ \frac{h(n)}{\log n}\geq\varepsilon>0\right\}$. There exists an $n_0^{(k)}\in\mathbb{N}$ for all $k\geq 2$ such that for all $n>n_0^{(k)}$ and $n\in A(\varepsilon)$ we have $h(n)\geq\varepsilon.\log n>k$ (it is sufficient to put $n_0^{(k)}=[e^{\frac{k}{\varepsilon}}]$, where $[x]$ is whole part of number x).

From this $A(\varepsilon)\cap\{n_0^{(k)}+1,n_0^{(k)}+2,\dots\}\subseteq\{n\in\mathbb{N}\ :\ h(n)\geq k\}$ for all $k\geq 2$, $k\in\mathbb{N}$.

Therefore $\sum_{n\in A(\varepsilon)}n^{-q}<+\infty$ for all $k\geq 2$ and $\mathcal{I}_c^{(q)}-\lim\frac{h(n)}{\log n}=0$ since the series (2) converges for all $q>\frac{1}{k}$. If $k\to\infty$ for sufficient large then $\mathcal{I}_c^{(q)}-\lim\frac{h(n)}{\log n}=0$ for all $q\in(0,1\rangle$. $\square$

Corollary 2.2. *We have*

$$\mathcal{I}_c^{*(q)}-\lim\frac{h(n)}{\log n}=0\ \textit{for all}\ q\in(0,1\rangle.$$

For the sequence $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$ we get the result of different character.

Theorem 2.3. *The sequence* $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*-convergent for every* $q\in(0,1)$.

Proof. In the paper [7] is proved, that the sequence $\left(\log p\cdot\frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is not $\mathcal{I}_c^{(q)}$-convergent to zero for any $q\in(0,1)$. The sequence $\left(\frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is also not $\mathcal{I}_c^{(q)}$-convergent to zero. The inequality $H(n)\geq a_p(n)$ holds for all $n=1,2,\dots$ and for any prime number p. Then we have $\frac{H(n)}{\log n}\geq\frac{a_p(n)}{\log n}$ for all $n=2,3,\dots$. This implies that the sequence $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$ is also not $\mathcal{I}_c^{(q)}$-convergent to zero for every $q\in(0,1)$. $\square$

Theorem 2.4. *For* $q=1$*, we obtain*

$$\mathcal{I}_c-\lim\frac{H(n)}{\log n}=0.$$

Proof. We will show that

$$A(\varepsilon) = \left\{ n \in \mathbb{N} \ : \ \frac{H(n)}{\log n} \geq \varepsilon \right\} \in \mathcal{I}_c$$

for any $\varepsilon > 0$.

Every non-negative integer n can be represented as $n = ab^2$, where a is a square-free number. Hence $H(a) = 1$ and

$$H(n) \in \{H(b^2), H(b^2) + 1\}.$$

If $n \in A(\varepsilon)$ then from $H(n) \geq \varepsilon \log n$ we have

$$\log(ab^2) \leq \frac{H(b^2) + 1}{\varepsilon} \qquad \text{and so} \qquad \log a \leq \frac{H(b^2) + 1}{\varepsilon}.$$

Therefore

$$A(\varepsilon) \subseteq B = \left\{ n \in \mathbb{N} \ : \ n = ab^2, \ \log a \leq \frac{H(b^2) + 1}{\varepsilon}, \ b \in \mathbb{N} \right\}.$$

It is enough to prove that $\sum_{n \in B} n^{-1} < +\infty$. We have

$$\sum_{n \in B} \frac{1}{n} = \sum_{b=1}^{\infty} \frac{1}{b^2} \sum_{\log a \leq \frac{H(b^2)+1}{\varepsilon}} \frac{1}{a}.$$

We use the inequality $S_k = \sum_{j=1}^{k} \frac{1}{j} \leq 1 + \log k$ for the harmonic series. Then we have the following inequality

$$\sum_{n \in B} \frac{1}{n} \leq \sum_{b=1}^{\infty} \frac{1}{b^2} \left(\frac{H(b^2) + 1}{\varepsilon} + 1 \right). \tag{3}$$

Because the $\sum \frac{1}{b^2} = \frac{\pi^2}{6} < +\infty$, it is enough to prove that the

$$\sum_{b=1}^{\infty} \frac{H(b^2)}{b^2} < +\infty. \tag{4}$$

For any $n \in \mathbb{N}$ we have $n = p_1^{a_1} \cdots p_k^{a_k} \geq 2^{H(n)}$ and from this $H(n) \leq \frac{\log n}{\log 2}$. Therefore

$$\sum_{b=1}^{\infty} \frac{H(b^2)}{b^2} \leq \frac{2}{\log 2} \sum_{b=1}^{\infty} \frac{\log b}{b^2} < +\infty.$$

We have shown that the sum in (4) is finite and therefore the sum in (3) is also finite.

Moreover $B \in \mathcal{I}$ and because $A(\varepsilon) \subseteq B$ we have $A(\varepsilon) \in \mathcal{I}_c$. □

The situation for sequences $\left(\frac{\omega(n)}{\log\log n}\right)_{n=2}^{\infty}$ and $\left(\frac{\Omega(n)}{\log\log n}\right)_{n=2}^{\infty}$ is different.

Theorem 2.5. *The sequences* $\left(\frac{\omega(n)}{\log\log n}\right)_{n=2}^{\infty}$ *and* $\left(\frac{\Omega(n)}{\log\log n}\right)_{n=2}^{\infty}$ *are not* $\mathcal{I}_c^{(q)}$*-convergent for all* $q \in (0,1\rangle$.

Proof. We prove this assertion only for $\left(\frac{\omega(n)}{\log\log n}\right)_{n=2}^{\infty}$. The proof for the sequence $\left(\frac{\Omega(n)}{\log\log n}\right)_{n=2}^{\infty}$ is analogous. Let $q = 1$. On the basis of the Theorem 2.2 of [23] and Lemma 1.6 we can assume that $\mathcal{I}_c - \lim \frac{\omega(n)}{\log\log n} = 1$. Take $\varepsilon \in \left(0, \frac{1}{2}\right)$ and consider the set

$$A(\varepsilon) = \left\{ n \in \mathbb{N} \ : \ \left| \frac{\omega(n)}{\log\log n} - 1 \right| \geq \varepsilon \right\}.$$

Put $n = p$ where p is a prime number, then $\omega(p) = 1$ and $\left|\frac{1}{\log\log p} - 1\right| \geq \varepsilon$ holds for all prime numbers $p > p_0$. Therefore the set A_ε contains all prime numbers greater than p_0. For these p we have: $\sum_{p>p_0} \frac{1}{p} = +\infty$ and so $A(\varepsilon) \notin \mathcal{I}_c$. From this $\mathcal{I}_c - \lim \frac{\omega(n)}{\log\log n} \neq 1$. Under the inclusion $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_c^{(1)} \equiv \mathcal{I}_c$ and according to Lemma 1.6 we have $\mathcal{I}_c^{(q)} - \lim \frac{\omega(n)}{\log\log n} \neq 1$ for $q \in (0,1\rangle$. This complete the proof. $\square$

Similar results we can prove for functions $f(n)$ and $f^*(n)$.

Theorem 2.6. *The sequence* $\left(\frac{\log\log f(n)}{\log\log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*-convergent for all* $q \in (0,1\rangle$.

Proof. According to Theorem 2.1 of [21] suppose that the

$$\mathcal{I}_c^{(q)} - \lim \frac{\log\log f(n)}{\log\log n} = 1 + \log 2,$$

where $q \in (0,1\rangle$. Let $\varepsilon \in (0, \log 2)$ and define the set

$$A(\varepsilon) = \left\{ n \in \mathbb{N} \ : \ \left| \frac{\log\log f(n)}{\log\log n} - (1 + \log 2) \right| \geq \varepsilon \right\}.$$

Put $n = p$, where p is a prime number, then $f(p) = p$ and $\frac{\log\log p}{\log\log p} = 1$. Therefore the set $A(\varepsilon)$ contains all prime numbers. Next we have:

$$\sum_{n\in A(\varepsilon)} n^{-q} \geq \sum_{j=1}^{\infty} {p_j}^{-q} \geq \sum_{j=1}^{\infty} {p_j}^{-1} = +\infty, \quad q \in (0,1\rangle.$$

Hence $A(\varepsilon) \notin \mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)} - \lim \frac{\log\log f(n)}{\log\log n} \neq 1 + \log 2$ for all $q \in (0,1\rangle$. □

Theorem 2.7. *The sequence* $\left(\frac{\log\log f^*(n)}{\log\log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*-convergent for all* $q \in (0,1\rangle$.

Proof. According to Theorem 2.2 of [21] again suppose that the

$$\mathcal{I}_c^{(q)} - \lim \frac{\log\log f^*(n)}{\log\log n} = 1 + \log 2,$$

where $q \in (0,1\rangle$. The proof is going similar as in the previous Theorem. Put $n = p_i p_j$, $i \neq j$, where p_i, p_j are distinct prime numbers. Then $f^*(n) = f^*(p_i p_j) = \frac{f(p_i p_j)}{p_i p_j} = \frac{p_i p_j (p_i p_j)}{p_i p_j} = p_i p_j$, $i \neq j$. Hence $\frac{\log\log f^*(p_i p_j)}{\log\log p_i p_j} = 1$. Let $\varepsilon \in (0, \log 2)$ and define the set

$$A(\varepsilon) = \left\{ n \in \mathbb{N} \ : \ \left| \frac{\log\log f^*(n)}{\log\log n} - (1 + \log 2) \right| \geq \varepsilon \right\}.$$

This set contains all numbers of the type $p_i p_j$, $i \neq j$. For $q \in (0,1\rangle$ we have:

$$\sum_{n\in A(\varepsilon)} n^{-q} \geq \sum_{\substack{j=1 \\ p_j\neq 2}}^{\infty} \frac{1}{2p_j}, \quad (p_i = 2).$$

Since the series $\sum_{j=1}^{\infty} \frac{1}{2p_j}$ diverges, we have $A(\varepsilon) \notin \mathcal{I}_c^{(q)}$ for all $q \in (0,1\rangle$. Therefore $\mathcal{I}_c^{(q)} - \lim \frac{\log\log f^*(n)}{\log\log n} \neq 1 + \log 2$ and the proof is complete. □

There exists a relationship between functions $f(n)$ and $\tau(n)$, where $\tau(n)$ is the number of divisors of n. The following equality holds: $\log f(n) = \frac{\tau(n)}{2}.\log n$, $(n > e)$ (see [12]). From this we have

$$\log\log f(n) = \log\frac{1}{2} + \log\tau(n) + \log\log n, \quad n > e^e.$$

Therefore

$$\frac{\log\log f(n)}{\log\log n} = 1 + \frac{\log\tau(n)}{\log\log n} + \frac{\log\frac{1}{2}}{\log\log n}, \quad n > e^e.$$

From Theorem 2.6 we have the following statement.

Corollary 2.8. *The sequence* $\left(\frac{\log \tau(n)}{\log\log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*-convergent for all* $q \in (0,1\rangle$.

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