Modelling of Ball and Plate System Based on First Principle Model and Optimal Control

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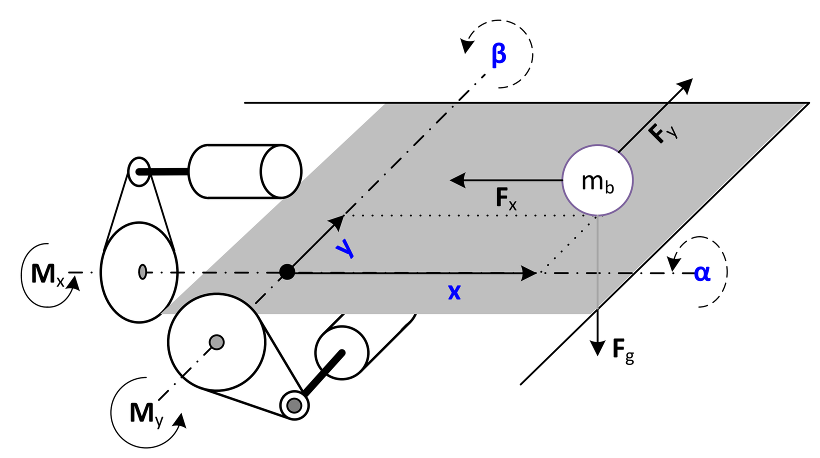
- **Topic** – ball and plate system modelling and optimal control
- **Task** – trajectory tracking of ball and plate system
- **Solution** – modelling of ball and plate system based on first principles by considering balance of forces and torques – by Newton-Euler method. A non-linear model is derived considering the dynamics of motors, gears and ball and plate. The non-linear model is linearized near the operating region to obtain a standard state space model. The linear model is used for discrete optimal control design – ball is tracked by control voltages of the motors.

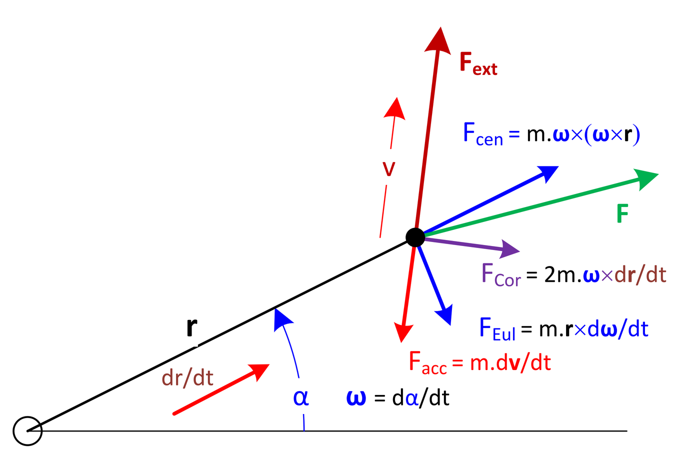
MODELLING

(for one axis)

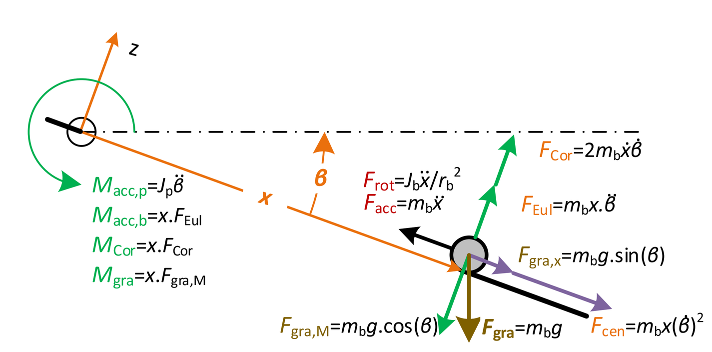


Balance of forces - ball $F_{acc} + F_{rot} + F_{los} + F_{Eul} + F_{Cor} + F_{cen} = F_{gra}(\alpha, \beta)$

$$\left(1 + \frac{J_b}{m_b r_b^2}\right)\frac{d^2x}{dt^2} + \frac{dx}{dt}\left(\frac{k_b}{m_b r_b} + \frac{k_{cx}}{m_b}\left|\frac{dx}{dt}\right|\right) + x\left(\frac{d\beta}{dt}\right)^2 = -g.\sin(\beta)$$



Forces acting on ball and plate



Movement in the x-axis (x-z plane) and rotation (torque) acting in y-axis

Balance of moments - plate with ball

$$M_{acc,p} + M_{acc,b} + M_{Cor,b} + M_{los} = M_{mot} - M_{gra}$$

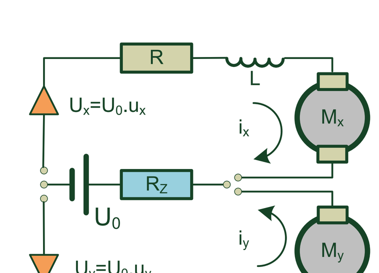
$$\left(\frac{J_p + J_{Gx}}{m_b} + y^2\right)\frac{d^2\alpha}{dt^2} + \left(2y\frac{dy}{dt} + \frac{k_{px}}{m_b}\right)\frac{d\alpha}{dt} = \frac{M_{Gx}}{m_b} - y.g\cos(\alpha)$$

Gear system $\alpha = \frac{1}{G}\varphi_x, \qquad M_{Gx} = G.M_x$

Balance of energy and moment – motor

$$R.i_x + L\frac{di_x}{dt} + k_u\frac{d\varphi_x}{dt} + R_z(i_x + i_y) = u_x.U_0$$

$$J_m\frac{d^2\varphi_x}{dt^2} + k_o\frac{d\varphi_x}{dt} + M_x = k_m.i_x$$



Equivalent circuit of DC motor

Final model

$$a_1\frac{d^2x}{dt^2} + \frac{dx}{dt}\left(a_2 + a_3\left|\frac{dx}{dt}\right|\right) + x\left(\frac{d\beta}{dt}\right)^2 = -g.\sin(\beta)$$

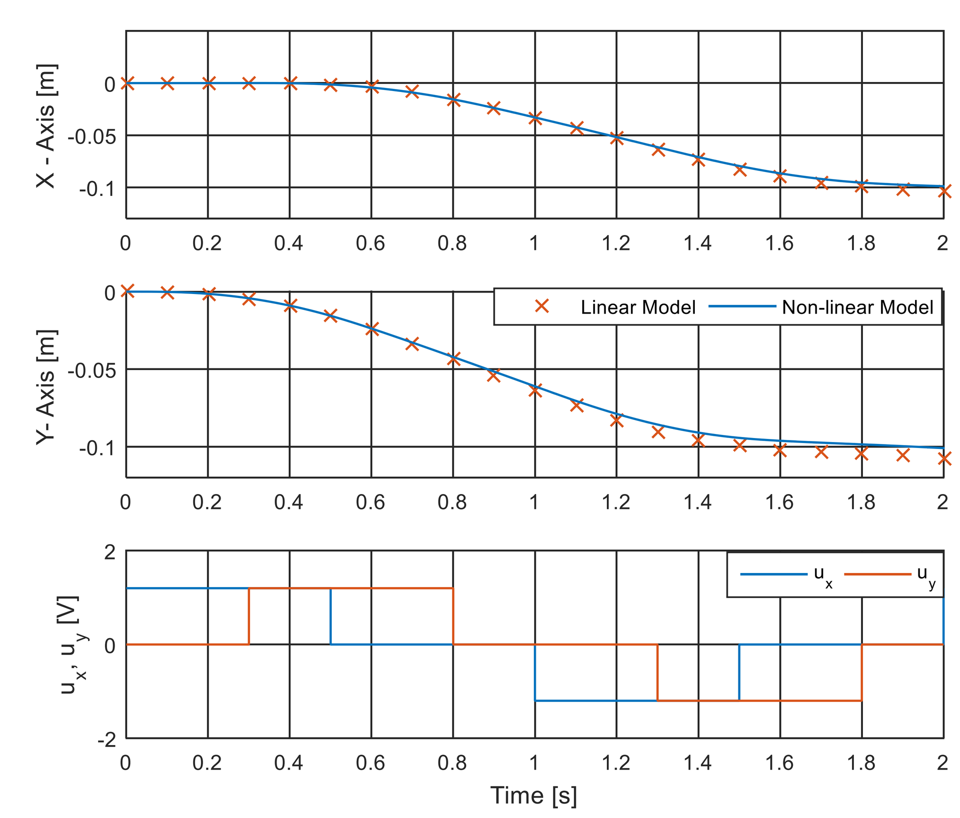
$$(b_{1x} + y^2)\frac{d^2\alpha}{dt^2} + \left(2y\frac{dy}{dt} + b_{2x}\right)\frac{d\alpha}{dt} = b_3 i_x - y.g\cos(\alpha)$$

$$\frac{di_x}{dt} + d_3\frac{d\alpha}{dt} + d_2 i_x + d_1 i_y = d_0 u_x$$

LINEAR STATE SPACE MODEL

$$\frac{d\bar{\mathbf{x}}}{dt} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u}$$
$$\mathbf{y} = \bar{\mathbf{C}}\bar{\mathbf{x}}$$

$$\bar{\mathbf{x}} = [\ x \quad v_x \quad \alpha \quad \omega_x \quad i_x \quad y \quad v_y \quad \beta \quad \omega_y \quad i_y\ ]^T$$



Outputs and inputs in open loop verification

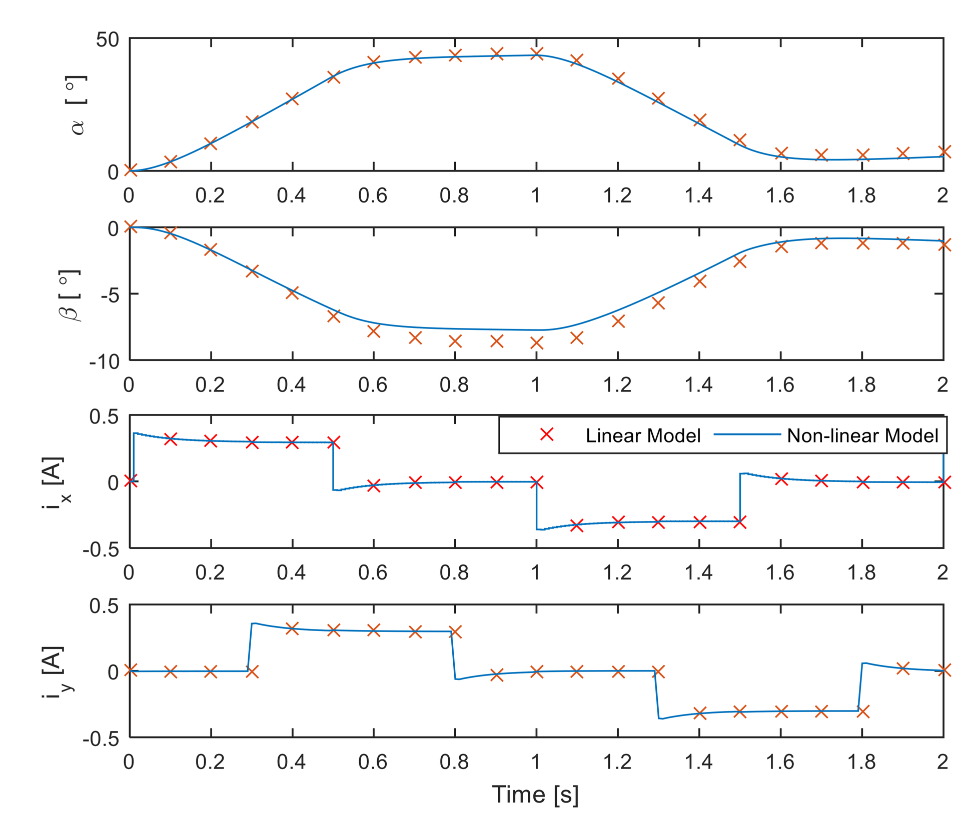


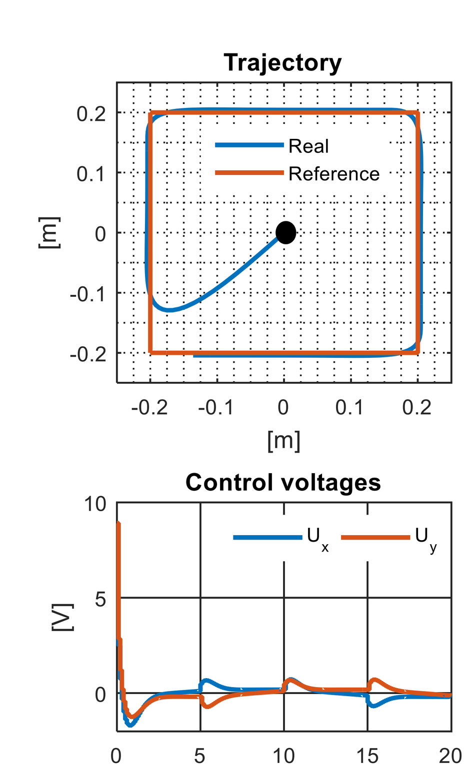
Plate angles and motor currents in open loop verification

OPTIMAL CONTROL

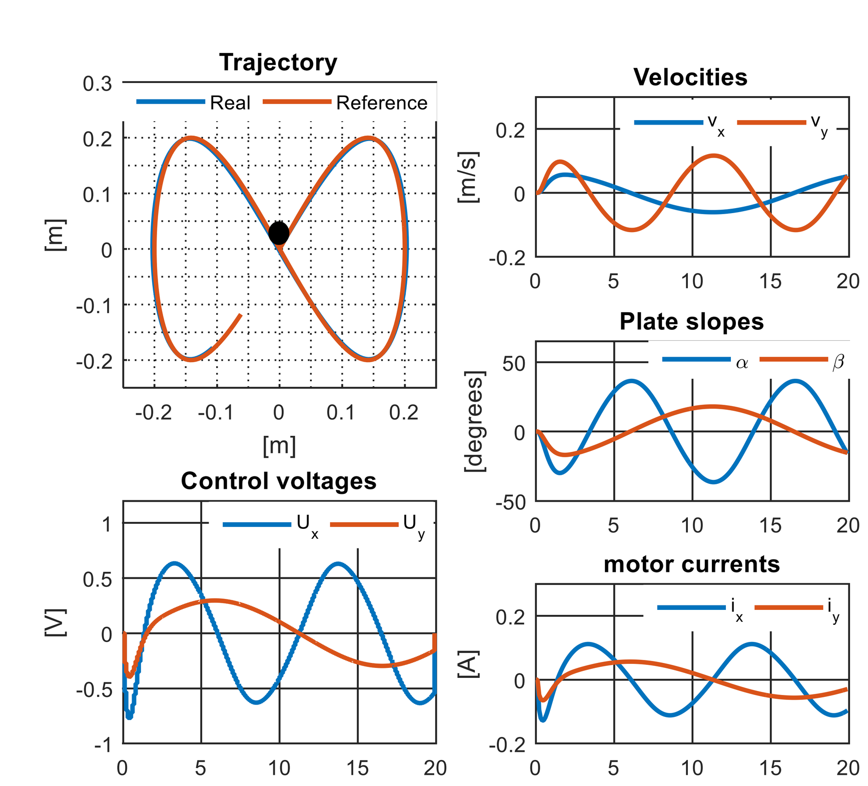
$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$
$$\boldsymbol{y}(k) = \mathbf{C}\mathbf{x}(k)$$

$$J_\infty = \sum_{i=1}^{\infty}[\mathbf{x}^T(k+i)\mathbf{Q}.\mathbf{x}(k+i) + \mathbf{u}^T(k+i)\mathbf{R}.\mathbf{u}(k+i)]$$

$$\mathbf{u}(k) = -\mathbf{K}[\mathbf{x}(k) - \mathbf{x_w}(k)]$$



Square trajectory tracking



Lissajous curve trajectory tracking