SCIENTIFIC PAPERS
OF THE UNIVERSITY OF PARDUBICE
Series B
The Jan Perner Transport Faculty
12 (2006)

A MATHEMATICAL MODEL FOR ELEMENTS ALLOCATION

Vlastislav MOJŽÍŠ, Tatiana MOLKOVÁ

Department of Transport Technology and Control

1. Introduction

One of the problems in transport systems is allocation of elements. An element can be a railway carriage, semi-trailer, container, pallet, etc. A suitable tool to solve the optimization task is a mathematic model complemented by logical operations. A linear dynamic model is used in the paper, which means linear dependence both in the objective function, and in the limiting conditions.

2. Requirements for solving of the problem

- optimization of elements allocation and maximum coverage of their destinations,
- compliance with the strategy of elements allocation,
- taking in regard the elements structure and their possible substitution,
- concentration of elements allocation control into minimized number of locations,
- dynamic concept,
- compliance with the transport network permeability,
- reasonable time for processing.

3. Evaluation criteria

The objective of the task is to minimize the total cost raised from partial costs of the elements relocation. Partial costs in the model are expressed as an arithmetic product of the evaluation tariffs and the number of relocated elements from the source node to the destination node. Numbers of elements are sought for in solving of the problem, tariffs must be determined prior to the calculation.

It is essential to reflect the complex costs for an element in the tariff, i.e. at all stages of the relocation procedure from the source to the destination node that is to include the following costs into the tariff:

- costs relating to the stay of an element at the source node and in the node of its destination,
- costs relating to the staging nodes service,
- costs relating to an element relocation in the network, i.e. between the source node and the destination node.

Tariffs can be expressed in nominal or natural form, the former being expressed in monetary units (CZK, €), determined by a mere aggregate sum of the costs in individual stages of the elements allocation procedure. The latter transfers the costs of each stage into kilometres, with the aim to simplify the whole procedure. For example, the costs of service at staging nodes are transferred according to their significance and reflect their presumable additional distance to be travelled.

Another solution makes use of the tariff expressed in kilometres. The background of this solution is an O/D matrix [1]. By a modification of such tariff (e.g. by means of a coefficient) the following results can be achieved, among others:

- taking into account the character of a network used for the element relocation (main or secondary road),
- expression of real relocations (in terms of technology, time aspect),
- convenience of an element substitution by a different type, and other impacts.

Evaluation criteria can be altered (e.g. in case of emergency). Taking into account the kilometre-based tariffs, the subject of minimizing in the objective function is the distance in km, for which the elements are relocated (further referred to as “Ekm”). The quality of the individual options is evaluated on the basis of the overall minimization result expressed in kilometres [3].

4. Mathematical formulation

A network (or a part thereof) has m numbers of node characterized by the source of elements a_1 a_2, … a_m and n nodes b_1, b_2,… b_n, needed for loading where:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{1}$$

Next, the tariff matrix m. n of numbers s_{ij} is determined.

The task is to find the number of elements to be relocated from the source a_i to the nodes with the destination of elements b_j, where:

$$c_{ij} \geq 0 \qquad for\ i = 1,2,...m;\ j = 1,2,...n$$

$$\sum_{j=1}^{n} c_{ij} = a_i \quad for\ i = 1,2,...m \tag{2}$$

$$\sum_{i=1}^{m} c_{ij} = b_j \qquad for\ i = 1,2,...n \tag{3}$$

and objective function F

$$F = \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} c_{ij} \tag{4}$$

will be minimized.

In the case that the problem is not balanced (i.e. the relation 1 does not hold), it can be converted using a fictive source of elements node a_{m+1}:

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \tag{5}$$

if

$$\sum_{i=1}^{m} a_i \langle \sum_{j=1}^{n} b_j \tag{6}$$

or using a fictive destination node of elements b_{n+1}:

$$b_{n+1} = \sum_{i+1}^{m} a_i - \sum_{j=1}^{n} b_j \tag{7}$$

if

$$\sum_{i=1}^{m} a_i \rangle \sum_{j=1}^{n} b_j \tag{8}$$

The above mentioned examples present a simple transport problem formulation by means of linear programming. A modified minimum cost circulation setup is such that the method of graph theory can be used.

5. Modified method of minimum cost circulations setup used for elements allocation

The method is firstly applied on a simple basic model that is gradually extended by further requirements.

5.1 Basic model

The model is based on the following assumptions:

- Ekm is considered to be the criterion for optimization,
- the fleet is made by only one type of fully replaceable elements,
- time is expressed in time horizon,
- at each relation, the time needed for the relocation is known with one period accuracy,
- minimum time for relocation is one period,
- the nodes in the network (e.g. representing a group of nodes) have non-negative source, that is non-negative numbers of elements and, at the same time, non-negative destinations known for each period up to the forecast period;
- lower and upper flow limits for permissible relocations are determined between some pairs of nodes.

The time span between making two consequent elements allocation decisions is understood as the period. A period can be a day, a working shift or a shorter period.

Time horizon is expressed by a multiple of periods needed for optional realization of the least favourable permissible relocation. Further, three periods are expected, called the 1st, the 2nd and the 3rd period.

Sources and destinations of elements in the nodes assign time in each period. In that model there are used for the solution sources of elements in the 1st and the 2nd period, their destinations in the 2nd and the 3rd period. From sources of elements in the 1st period can meet the destinations in the 2nd and the 3rd period, from sources in the 2nd period only destinations in the 3rd period.

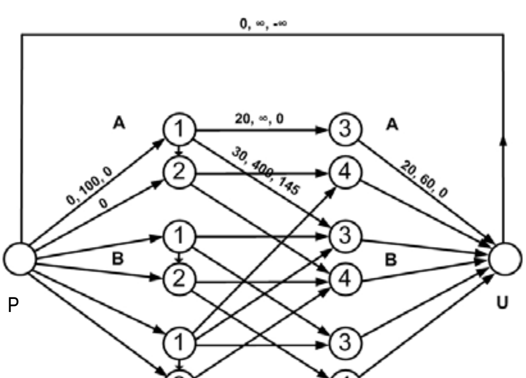

Fig. 1 *Base network*

The application is carried out on the example with three nodes. The application is shown in Fig. 1. It demonstrates a modified transport network with three nodes A, B, C. The node springs (sources element nodes) are on the left, and mouths (nodes with destination of elements) are on the right. To simplify the algorithm, the network is complemented by pseudo-spring P and pseudo-mouth U. The indexes at nodes A, B, C peaks show the periods, namely:

1 – the first period
2 – the second period } source element nodes,

3 – the second period
4 – the third period } nodes with the destination of elements.

The edges between the original springs and mouths represent the permissible relocations reflecting the time and technology point of view. Three input data values will be allocated to the edges, indicated in Fig. 1, respectively:

d – lower limit of elements flow,

h – upper limit of elements flow,

s – unit tariff for one edge element relocation.

The input data are also assigned to the edges connecting the pseudo-spring with the pseudo-edges, their value is, however, different at the following items:

- lower limit of elements flow is equal to 0,
- upper limit of elements flow equals to the source vehicles at given spring according to individual periods,
- at unit tariff it is equal to 0.

The same applies at the edges connecting the mouth with pseudo-edges, except when the upper limit of elements flow equals to the destinations of elements in given mouth according to individual periods.

To provide for the circulation, pseudo-mouth U is joined with pseudo-spring P by means of a backward edge, allocated by the following values: d = 0, h = ∞, s = - ∞, which enables to achieve maximum flow in the network. Figure 1 demonstrates the values of d,h and s data as an illustration of some edges.

By means of a different limitation of the flow at the edges between the springs and mouths starting at 0, we can ensure, among others, the priority satisfaction of elements destinations (e.g. 30 elements in node B in the second periods – edge A1, B3), or a minimum number of elements in the relation regarding the appropriateness turnover, etc. Upper flows limit at these edges can limit the number of elements – carriages – in regard to the network permeability.

Linking the peak and index 1, that has index 2 in the same source element node expresses the option of intentional hold-back of an element in the first period and its dispatching in the second period, or optional exceeding of the service time from the first period.

At the same time, the nodes indicate the elements sources in the 1 and 2 periods, expressing the requirements for the 2 and 3 periods. This can be expressed by the edges between the springs and mouths of the same node. In case that a node represents a group of nodes in a network, and an element is within such group required at a different location, such fact is expressed by a non-zero unit tariff (edges C2, C4). If the same location is in question, the tariff can be zero (edges A1, A3).

The edges C1, A4 represent a situation when relocation of source elements to the destinations node can be carried out for two periods only. Non-permissible relocation (e.g. due to technology or time constraints) is expressed by the fact that an edge between such sources node and destinations node is missing (e.g. between nodes A and C).

A solution for given network is sought for by means of the Out of Kilter algorithm [3], [4]. Such a solution should:

- show the smallest overall Ekm value (optimal solution);
- determine the numbers of relocated elements between individual sources nodes and destinations nodes;
- satisfy the overall elements destinations (maximizing network circulation) to the greatest extent;
- satisfy the priority destinations of elements (by means of non-zero lower flow limits) to maximum extent.

The following requirements have been complied with in the basic model (see Chapter 2):

- optimization,
- dynamic concept,
- track permeability.
- priority destinations satisfaction – within the strategy.

5.2 Extension of basic model

5.2.1 Types of elements and their replacement

The assumptions for the initial model construction will be extended by designing two types of partially replaceable elements (as per the commodity type). Other assumptions will be maintained.

The compliance with such requirement is illustrated in Fig. 2 in a network fragment. For a clear demonstration, only one pair of nodes has been used.

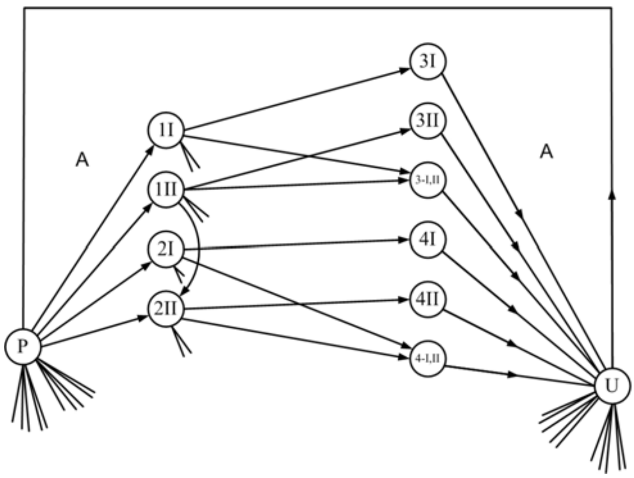

Fig. 2 *Fragment of extended network*

Two types of elements are marked by Roman numerals I and II. Marking of nodes has been modified in Fig. 2, representing the following items:

1 I - source of elements type I in the 1st period

1 II - source of elements type II in the 1st period

2 I - source of elements type I in the 2nd period

2 II - source of elements type II in the 2nd period

3 I - destination of elements type I in the 2nd period

3 II - destination of elements type II in the 2nd period

3 I, II - destination of elements type I or type II in the 2nd period

4 I - destination of elements type I in the 3rd period

4 II - destination of elements type II in the 3rd period

4 I, II - destination of elements type I or II in the 3rd period.

Possible reciprocal exchange of elements during their destination is expressed by nodes marked 3 - I, II and 4-I, II.

As in the basic model, the data values d,h, and s are assigned to the individual edges. Further procedure is equal with that of the basic model.

Obviously, more element types can be used in the model, however, the network would be more complicated. Given method of taking the elements structure into account and their optional replacement can be further simplified. Such a simplification is carried out together with the alterations relating to the nodes in the following chapter (part II).

5.2.2 Modifications relating to nodes

The meaning of modifications relating to nodes is the integration of further requirements for the improvement at parallel simplification of the model. The objective of such adjustments is the creation of assumptions for the smallest number of operation places.

In the transport network of an initial model, one node can represent a group of a certain type nodes (super-node). The source vehicle nodes are usually linked directly by edges with the destination nodes.

Regarding the formation of units, the nodes cannot be directly interlinked. Such state does not exist among all the super-nodes. The model offers two solutions:

I. Excluding super-nodes detachment

The nodes are divided into main and subsidiary nodes. The main nodes can dispatch elements to other main nodes as well as into near subsidiary nodes, but the subsidiary ones only to neighbouring main nodes and near subsidiary nodes. In elements relocation between the destination nodes, we can expect a time less then one period. Connecting main nodes with subsidiary nodes in the same period (the same index) presents the expression of given state in the network.

The following model integrates all the assumptions of the two previous models. Only one type of elements is used for clear demonstration, and only several edges. Illustration of such model is in Fig. 3. The main nodes are marked A, B and C, the subsidiary ones D and E.

For illustration, only the edges of B node into C node are drawn, out of those connecting the nodes of different main nodes. The option of mutual relocation of elements between the main and subsidiary nodes in the same period is illustrated by arc edges.

This modification enables to concentrate the element flow into edges connecting the main nodes, reducing the overall number of edges (with favourable impact on the calculation time), as well as easier processing of the units formation schedule.

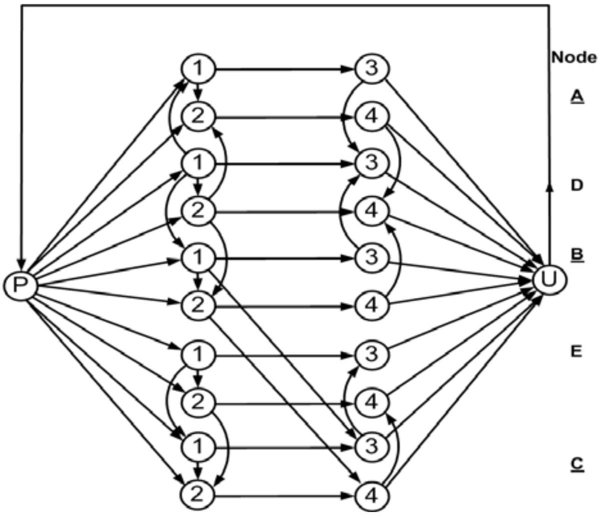

Fig. 3 *Network with main and subsidiary nodes*

II. Including super-nodes detachment

This modification relating to the nodes can express – in comparison with I. – the option of staging service of elements in super-nodes. A simplified example is given in Fig. 4.

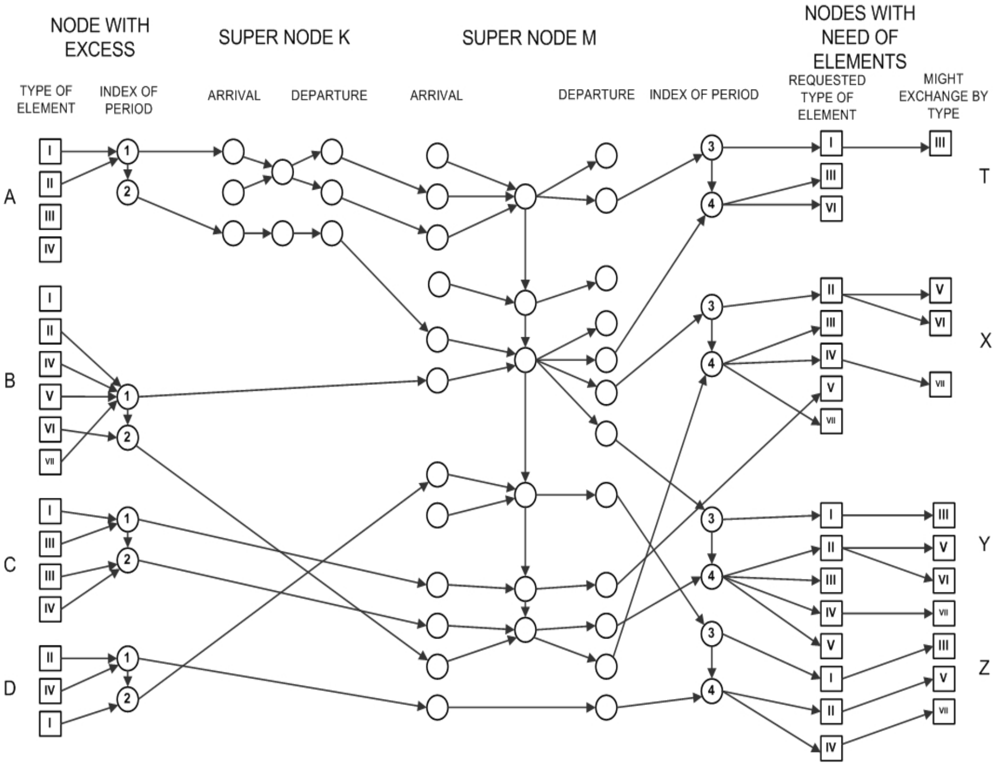

Fig. 4 *Transport network with super-nodes*

The network fragment shows the source side on the left and the destination nodes on the right. Nodes A, B, C, D on the source side are connected with the pseudo-spring P. Types of elements are marked by a Roman number in cubic peaks, and the circled number indicates the period of time (1 and 2 period), showing the elements ready to relocation at certain time.

The situation on the destination side is similar. Nodes T, X, Y and Z are connected to the mouth U (not illustrated in the figure). The option of elements replacement is illustrated by the connection of edges in the first column of cubic peaks with the second column. Circular peaks mark similar periods of time at the arrival of the required elements.

The peaks are gradually located between the source and destination sides within individual super-node, which expresses the option (or destination) of staging service. The division of peaks in each serviced super-node into three columns according to time (arrival – service – departure) expresses the transfer of elements from one set to another. At the super-node serviced as free from the elements transfer from one set to another, only the peaks arrival and departure are sufficient, (relation D-Z), or only one peak, if needed. A source node is directly connected with a destination node, unless the elements of a certain relation do not cross the super-node.

The source nodes are connected to the peaks in a super-node by edges type node - unit. In a serviced super-node, the peaks on the arrival side (one for each set) connect by the edges between the serviced nodes (for a simple graphic illustration, the transfer of elements from one set to another), and further by the edges type peak – peak with the peaks on the departure side.

Departure node for each set is connected by an edge type unit – node with the destination node side in an appropriate period. Super-nodes are interlinked by the unit type edges. Data d, f and h are assigned to the edges. Looking for the minimum cost flows in the form of circulation is enabled by the backward edge from pseudo-mouth U into pseudo-spring P (not illustrated in Fig. 4).

All the requirements are completed using the method of circulation setup with minimum costs.

6. Conclusion

Allocation of elements is usually dealt with by means of a linear programming mathematic model, i.e. using the transport problem. This method does not work with the requirements for a dynamic concept (time factor, exchange of elements or their formation). Such drawbacks can be improved by means of the graph theory, especially by circulation setup with minimized costs and using the Out of Kilter algorithm.

REFERENCES

1. STEENBRINK, P.: *Optimization of transport network*, : John Wiley & Sons, London, 1974, ISBN 0-7546-1279-1
2. CASCETTA, E: *Transportation systems engineering: Theory and Methods*, Kluwer Academic Publisher, Dordrecht, 2001, ISBN 0-7546-1279-1
3. HENSHER, D. A., BUTTON, K. J.: *Handbook of transport systems and traffic control*, Pergamon Press, Amsterdam 2001, ISBN 0-08-043595-5
4. FORD, L. R., FULKERSON, D. R.: *Flows in Networks*, Princeton University Press, 1962

The paper is supported by the Czech Scientific Foundation GA CR, grant project No.103/05/2043.

Lektoroval: doc. RNDr. Antonín Tuzar, CSc.

Předloženo: 16.4.2007

Summary

A MATHEMATICAL MODEL FOR ELEMENTS ALLOCATION

Vlastislav MOJŽÍŠ, Tatiana MOLKOVÁ

In the paper a mathematical model for optimization of element allocation is presented. For the solution an advantageous modified method of framing circulations with minimal costs from graph theory was used. A generalized model was designed to comply with the requirements of given task.

Resumé

MATEMATICKÝ MODEL PŘIDĚLOVÁNÍ ELEMENTŮ

Vlastislav MOJŽÍŠ, Tatiana MOLKOVÁ

Příspěvek se zabývá problematikou optimalizace přidělování elementů (vozů, výměnnných nástaveb, kontejnerů). Vhodným prostředkem k řešení této optimalizační úlohy je matematický model, doplněný logickými operacemi. Autoři k řešení použili lineární dynamický model, který obsahuje lineární závislosti jak v účelové funkci, tak i v omezujících podmínkách.

Zusammenfassung

MATHEMATISCHES MODELL FÜR DIE ELEMENTZUTEILUNG

Vlastislav MOJŽÍŠ, Tatiana MOLKOVÁ

Der Beitrag beschäftigt sich mit der Problematik der Optimierungsaufgabe Zuteilung der Güterwagen, Wechselbehälter, Container etc. Zur Lösung dieser Optimierungsaufgabe ist ein mathematisches Modell ergänzt um die logischen Funktionen zu verwenden. Zur Lösung verwanden die Autoren ein linear dynamisches Modell mit den Linearabhängigkeiten sowohl in der Zielfunktion, als auch in den Randbedingungen.