# THE DERIVATIVE AND MARGINAL COST, REVENUE, AND PROFIT

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***Abstract:*** *The article is focused to describe several examples of economical and business problems which are solved by using derivations. The examples are used in mathematical courses in the first year of study (the first term) at the Faculty of Economics and Administration.*

**Key words:** mathematics, derivate, economics, business, marginal, cost, revenue, profit

## 1 Introduction

In last few years there was a tendency to increase the quantity of mathematical courses and mathematical applied methods in all economic and IT fields of study at the Faculty of Economics and Administration.

The realization of this intention is made by introduction of applied mathematical methods into special subjects in later years of study. This process must be supported at the beginning of the study particularly in mathematical courses Mathematics I and Mathematics II in the first year.

It means that in those subjects is intention to apply mathematical examples at the highest possible level. Those examples are orientated mainly to economic, business and manufacture problems.

From academic year 2003/2004 teaching hours of mathematical seminars in Mathematics I and II courses increased from 2 to 3 hours per week. And thus, more mathematic applications should be realized in those subjects.

The aim of the article is to give non-economist teachers of mathematic information about basic economic variables which are someway in relation with derivation. There is an attempt to show few examples where one variable derivation is used (see [2], [3]).

## 2 Marginal cost

In business and economics, the rates at which certain quantities are changing often provide useful insight into various economic systems. A manufacturer, for example, is interested not only in the total cost $C(x)$ at certain production levels, but is also interested in the rate of change of costs at various production levels.

In economics, the word marginal refers to a rate of change (see [1]); that is, to a derivative. Thus, if $C(x)$ is the total cost of producing $x$ items then $C'(x)$ is the marginal cost, i.e. the instantaneous rate of change of total cost $C(x)$ which respect to the number of items produced at a production level of $x$ items.

### Example 1

Suppose the total cost $C(x)$ (in millions of euros) for manufacturing $x$ air-planes per year is given by the function

$$C(x) = 6 + \sqrt{4x + 4} \qquad 0 \le x \le 30$$

a) Find the marginal cost at a production level of x air-planes per year.
b) Find the marginal cost at a production level of 15 and 24 air-planes per year, and interpret the results.

Solution

a) The marginal cost at a production level of x air-planes is

$$C'(x) = \left(6 + \sqrt{4x+4}\right)' = \frac{2}{\sqrt{4x+4}}$$

b) The marginal cost at a production level of 15 air-planes is

$$C'(15) = \frac{2}{\sqrt{4(15)+4}} = 0.25$$

At a production level of 150 air-planes per year, the total cost is increasing at the rate of €250,000 per one air-plane.

The marginal cost at a production level of 24 air-planes is

$$C'(24) = \frac{2}{\sqrt{4(24)+4}} = 0.2$$

At a production level of 150 air-planes per year, the total cost is increasing at the rate of €200,000 per one air-plane.

**Exact cost**

If $C(x)$ is the total cost of producing x items, then the exact cost of producing the $(x+1)^{st}$ item is

$$C(x+1) - C(x).$$

The marginal cost function approximates exact cost of producing the $(x+1)^{st}$ item

$$C'(x) \approx C(x+1) - C(x).$$

**Example 2**

The total cost function (in thousands of euros) for manufacturing $x$ manipulators per year is given

$$C(x) = 375 + 25x - 0.25x^2 \qquad 0 \le x \le 50$$

a) Use the marginal cost function to approximate the cost of producing the $31^{st}$ manipulator.
b) Use the total cost function to find the exact cost of producing the $31^{st}$ manipulator.

Solution

a) The marginal cost is $C'(x) = 25 - 0.5x$. Thus, $C'(30) = 25 - 0.5(30) = 10$.
The cost of producing of the $31^{st}$ manipulator is approximately €10,000.

b) The exact cost of producing the $31^{st}$ manipulator is

$$C(31) - C(30) = 909.75 - 900 = 9.75$$, i.e. €9,750.

The marginal cost of € 10,000 per manipulator is a close approximation to his exact cost.

**Marginal revenue**

The definition of the marginal revenue (according to [1]) is similar as the one of the marginal cost. If $R(x)$ is the total revenue of producing $x$ items then $R'(x)$ is the marginal revenue, i.e. instantaneous rate of change of total revenue $R(x)$ which respect to the number of items produced at a production level of $x$ items.

The exact revenue of producing the $(x+1)^{st}$ item is

$$R(x+1)-R(x).$$

The marginal revenue function approximates exact revenue of producing the $(x+1)^{st}$ item, i.e.

$$R'(x) \approx R(x+1)-R(x).$$

**Marginal profit**

Total profit is the difference between the total cost and total revenue

$$P(x)=C(x)-R(x),$$

and according to [1], the marginal profit is

$$P'(x)=C'(x)-R'(x).$$

The exact profit of producing the $(x+1)^{st}$ item is $P(x+1)-P(x)$. The marginal profit function approximates exact profit of producing the $(x+1)^{st}$ item, i.e.

$$R'(x) \approx R(x+1)-R(x).$$

**Example 3**

The market research department of a company recommends manufacture and market a new toy-car. The financial department provides the following cost function (in euros) $C(x)=7\,000+2\,x$; $0 \le x \le 10\,000$ where €7000 is the estimate of fixed costs and €2 is the estimate of variable cost per toy-car. The estimate of revenue function (in euros) is $R(x)=x\,(10-0.001\,x)$; $0 \le x \le 10\,000$.

a) Find the marginal cost function $C'(x)$ and interpret.
b) Find the marginal revenue function $R'(x)$ and interpret.
c) Find marginal revenue at $x=2\,000$, 5 000, and 7 000. Interpret these resultants.
d) Find the profit function $P(x)$ and marginal profit function $P'(x)$.
e) Find the marginal profit at $x=1\,000$, 4 000, and 6 000. Interpret these results.

Solution

a) The marginal cost is

$$C'(x)=2.$$

Since this is a constant, it cost an additional € 2 to produce one more toy-car at any production level.

b) The marginal revenue is

$$R'(x)=10-0.002\,x.$$

c) For production levels of $x=2\,000,\ 5\,000,$ and $7\,000,$ we have

$R'(2\,000)=6$ $\qquad$ $R'(5\,000)=0$ $\qquad$ $R'(7\,000)=-4$.

This means that at production levels of 2 000, 5 000, and 7 000, the respective approximate changes in revenue per unit change in production are €6, €0, and – €4. That is, at the 2 000 output level, revenue increases as production increases; at the 5 000 output level, revenue does not change with a “small” change in production; and at the 7 000 output level, revenue decreases with an increase in production.

d) The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.001\,x^2 + 8\,x - 7\,000 \end{aligned}$$

e) The marginal profit function is

$$P'(x) = -0.002\,x + 8\,.$$

For production levels of 1 000, 4 000, and 6 000, we have

$$P'(1\,000) = 6 \qquad P'(4\,000) = 0 \qquad P'(6\,000) = -4$$

This means that at production levels of 1000, 4000, and 6000, the respective approximate changes in profit per unit change in production are €6, €0, and – €4. That is, at the 1 000 output level, profit will be increased if production is increased; at the 4 000 output level, profit does not change for "small" changes in production; and at the 6 000 output level, profits will decrease if production is increased. It seems the best production level to produce a maximum profit is 4 000.

## 3 Conclusion

In examples used above are shown few applications of derivation of function in economic and business branches. Those examples could be involved to seminar of the Mathematic I course where derivation of function according to basic rules is practised.

Shown applied derivation examples could motivate students to study mathematics, as well as to study economic subjects. For understanding the basis level of economic relations is essential to understand derivations properly.

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