MODELLING TIME SERIES WITH CONDITIONAL HETEROSCEDASTICITY

The simple ARCH Model

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Over the last fifteen years, the interest in nonlinear time series models has been steadily increasing. Univariate time-series models may not work successfully if they restrict only to linear functions of past observations. The same past may well contain useful information for the present and future if the dependence is nonlinear. Among nonlinear functions we shall consider the simplest of the family of heteroscedastic models - the autoregressive conditional heteroscedastic or ARCH model that is based on the conditional variance structure. This model was first applied by Engle (1982) to estimate the variance of U.K. Inflation. The aim of this article is to find out whether ARCH models should also be applied to quarterly time series of the Portuguese Imports (escudos) for the period 1976 – 2004.

1. ARIMA models for the Portuguese Imports

ARIMA modelling of time series is based on weak stationarity which requires that, if the time series y_t is not constant in the mean and variance over time, some appropriate transformations can be performed in such a way to render that process stationary. These kind of models are then used to forecast future values of y_t , t =1, …, T, based on the conditional means of the series, implicitly assuming that the conditional variance remains constant. However, many economic time series (financial aggregates, interest rates, exchange rates, consumer price index and so on) do not have a constant mean and most of them exhibit phases of relative tranquillity followed by periods of high volatility.

Figure 1a shows that there is little point in modelling the quarterly time series for the Portuguese Imports as being stationary. There is positive trend. The first difference of the series presented in Figure 1b shows constant mean in the first part of the series although the end of the series suggests that the variance increases with time. Therefore, the logarithm of the imports (ln import_pt) series should be used to better capture the growth rates. This series shows almost constant trend. As shown in Figure1d, the first difference of the ln import_pt series is the most likely candidate to be covariance stationary. The augmented Dickey and Fuller test shows that the ln import_pt series is mean stationary (ADF = -4,50412 < 1% critical value = -0,35814). So, an appropriate ARIMA model can be applied to this series. Additional analysis will show whether an ARCH model should be better candidate for modelling the Portuguese Imports.

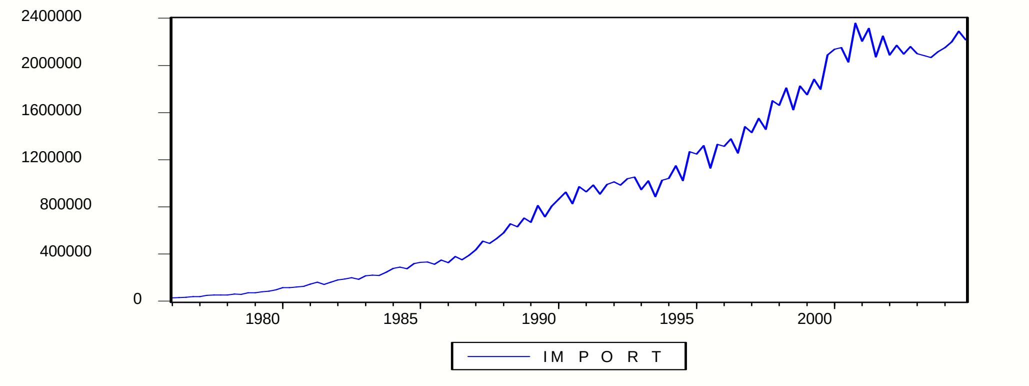


Figure 1a Series of the Portuguese Imports 1976 –2005 (Import_pt)

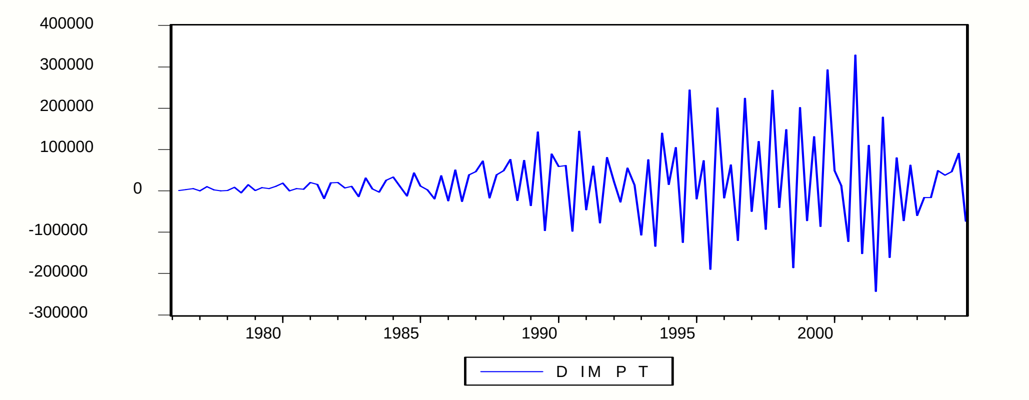


Figure 1b First difference series of the Portuguese Imports 1976 –2004

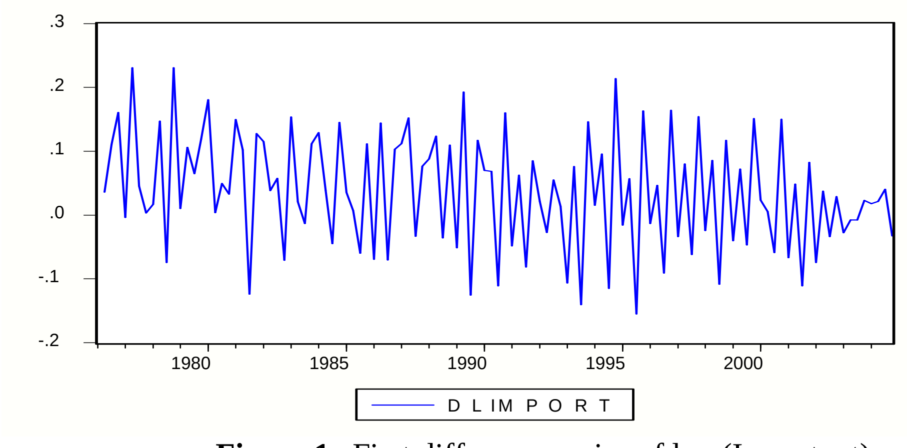
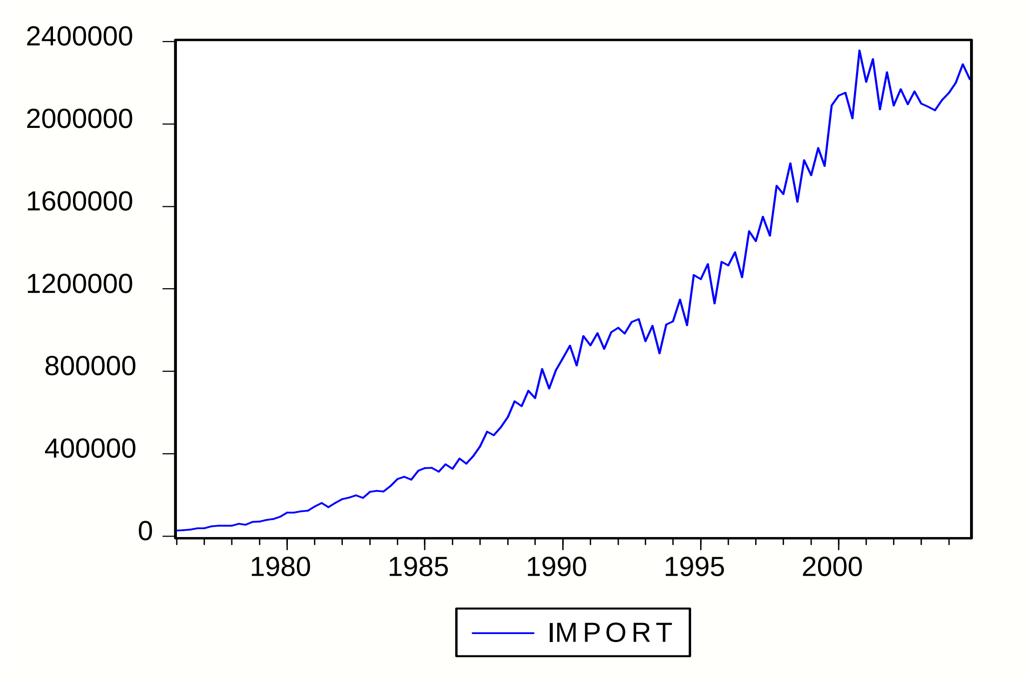


Figure 1c First difference series of log (Import_pt)

To forecast the quarterly Portuguese Imports for the period 1976– 2005, two ARIMA models – ARIMA(3, 1, 0) and ARIMA (0,1,3) – were identified. The ARIMA (3,1,0) was chosen taking into account the following analysis of the transformed time series:

- The ACF for (1-B) log (IMPORT $_t$) (see Fugure 2a) showed that only the third coefficient is statistically significant at the 5 percent significance level. Forcing the two first coefficients to be zero, the coefficient is: r_3 = 0,435 with se(r_3) = 0,133. Similar conclusion can be taken from the PACF of (1-B) log (IMPORT $_t$) series (see Figure 2b), in which ϕ_3 = 0,347 with standard error se(ϕ_3) = 0,141.

- After the estimation of both models, model comparison procedure, presented in Table 1, reveals that both models (A) = ARIMA(3,1,0) with constant and (B) = ARIMA(0,1,3) with constant are comparable in their standard errors of estimate (RMSE). The residuals of both models are not autocorrelated, but the residuals of model (B) are worse than the residuals of model (A) because they are not stable in their means.



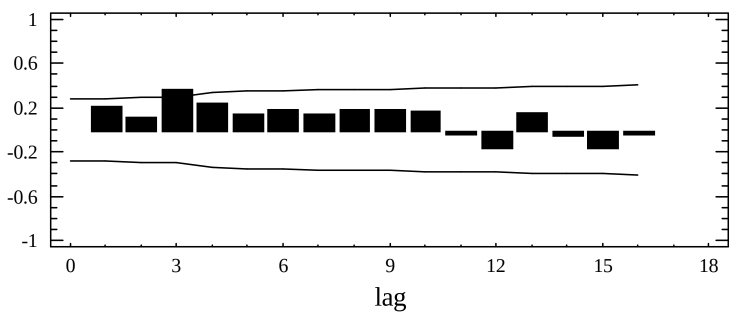


Figure 2a ACF for the first differences of the 1n(Import_pt) series

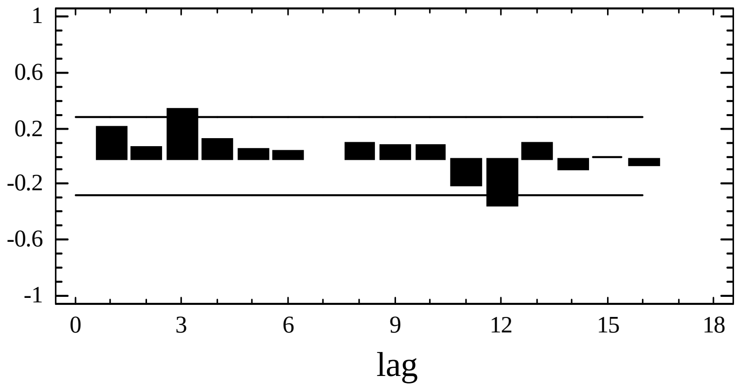


Figure 2b PACF for the first differences of the ln(Import_pt) series

Table 1 Diagnostic checks for model adequacy during the estimation period 1948–1996

Model	RMSE	MAPE	MPE	AUTO	MEAN	VAR
(A)	163940,0	9,33	–0,014	OK	OK	OK
(B)	161950,0	9,44	–0,196	OK	**	OK

Note: * = marginally significant (0.05 < p <= 0.10), ** = significant (0,01 < p ≤ 0,05) and

*** = highly significant (p ≤ 0,01).

AUTO = Box-Pierce test for excessive autocorrelation

MEAN = Test for difference in mean 1st half to 2nd half

VAR = Test for difference in variance 1st half to 2nd half

Figure 3 shows that, in spite of the fact that residuals are not correlated, they show high error variances, estimated by variance of residuals h_t = MSE = 2,68764E10 and standard deviation of residuals SE = 163 940.

Model (A) will be used to forecast the time series of the Portuguese Imports using a model of the form

$$(1-\phi_3 B^3)(1-B) \log (IMPORT_t) = K + a_t,$$

and its estimate

$$(1–0{,}435B^3)(1-B)\log (IMPORT_t) = 0{,}123 + a_t. \quad (1)$$

$$(0{,}133) \qquad\qquad (0{,}030)$$

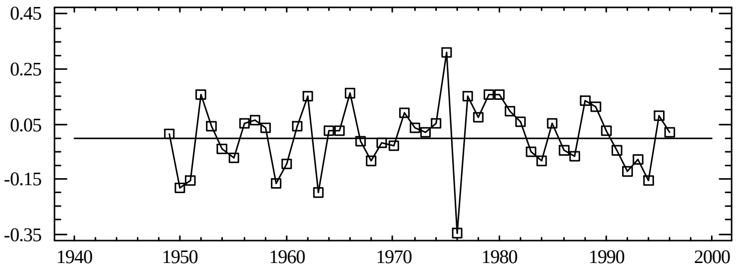


Figure 3 Residuals for ARIMA(3,1,0) model with constant

The estimated white noise variance is $\hat{\sigma}^2 = 0{,}0155468$ with 45 degrees of freedom, corresponding to an estimated white noise standard error of $\hat{\sigma} = 0{,}12469$. The Box-Ljung test using the 16 first autocorrelation coefficients of the errors rejects the null hypothesis of linear independence (Q (16) = 17,363, P=0,363, α =0,05).

Forecasts for the period 1996-2000 are presented in the Table 2.

Table 2 Forecasts of the Portuguese Imports using the estimated ARIMA (3,1,0) model

Period	Forecast	Lower 95,0% Limit	Upper 95,0% Limit	Reality	AE
1996	5,49533E6	4,28320E6	7,05049E6	5,427132E6	6,8198E4
1997	6,20796E6	4,23564E6	9,09869E6	6,139709E6	6,8251E4
1998	6,94066E6	4,25453E6	1,13227E7	6,914779E6	2,5887E4
2009	7,57214E6	3,96217E6	1,44712E7	7,519209E6	5,2931E4
2000	8,49296E6	3,83102E6	1,88280E7	8,672286E6	-17,9326E4

According to Table 1, it is expected that the Import forecasts will overestimate the reality about 9,33 %.

2. Autoregressive Conditional Heteroscedastic Model

The error variance for the ARIMA (3,1,0) model previously estimated is not constant. There is an autoregressive coefficient which can be the consequence of the ARCH effect in the errors as Weiss (1984) concluded in his work. What is the meaning of an ARCH effect?

According to Engle´s strategy, when the conditional variance is not constant, it is possible to model the conditional variance as an AR(q) process using the square of the estimated residuals obtained from the application of the ARIMA (3,1,0) model to the series y_t (the transformed Imports)

$$h_t = \hat{a}_t^2 = \alpha_0 + \alpha_1 \hat{a}_{t-1}^2 + \alpha_2 \hat{a}_{t-2}^2 + \ldots + \alpha_q \hat{a}_{t-q}^2 + \nu_t, \qquad (2)$$

where ν_t is a white-noise process.

Then, the best fitted ARIMA model for y_t together with model (2) is named an autoregressive conditional heteroscedastic model ARCH(q).

To test ARCH(q) effect in the time series, the correlogram should suggest such process. The technique is as follows:

Step 1. Estimate for the time series y_t the best-fitting ARIMA model (or regression model) and obtain the squares of the fitted errors $\hat{a}_t^2$. Also calculate the sample variance of residuals $\hat{\sigma}^2 = \sum_{t=1}^{T} \hat{a}_t^2 / T$, where T is the number of residuals.

Step 2. Calculate and plot the sample autocorrelation of the squared residuals as

$$r_k = \frac{\sum_{t=k+1}^{T} (\hat{a}_t^2 - \hat{\sigma}^2)(\hat{a}_{t-k}^2 - \hat{\sigma}^2)}{\sum_{t=1}^{T} (\hat{a}_t^2 - \hat{\sigma}^2)^2} \qquad (3)$$

Step 3. Test the hypothesis

H_0 : No ARCH(q) effect

H_1 : ARCH effect present.

There are several tests to take an appropriate decision:

- For large samples, the standard deviation of r_k can be approximated by $1/\sqrt{T}$. Individual values of r_k significantly different from zero at 5 % significance level are indicative of ARCH errors, if

$$| r_k | > 2/\sqrt{T} \qquad (4)$$

- Ljung-Box Q-statistics can be used to test for groups of the first *m* autocorrelation coefficients. In practice, we could consider values of *m* up to T/4.

The test statistic

$$Q = T(T+2)\sum_{k=1}^{m} r_k /(T-k) \qquad (5)$$

has an asymptotic χ^2 distribution with m degrees of freedom if the $\hat{a}_t^2$ are uncorrelated. For a given significance level α, the null hypothesis is rejected if

$Q > \chi^2_\alpha(m)$. Rejecting the null hypothesis that $\hat{a}_t^2$ are uncorrelated is equivalent to rejecting the null hypothesis of no ARCH errors.

- The more formal Lagrange multiplier test for ARCH disturbances was proposed by Engle (1982). The methodology differs from the previous one, that we regress squared residuals $\hat{a}_t^2$ on a constant and the q lagged values $\hat{a}_{t-1}^2$, $\hat{a}_{t-2}^2$, $\hat{a}_{t-3}^2$, ..., $\hat{a}_{t-q}^2$. That is, we will estimate the coefficients α_i of model (2) using OLS method. If there is no ARCH effect, the values of α_i for i = 1, ..., *q* should be zero. Hence, this regression will have little explanatory power so that the coefficient of deremnination (i.e. the usual R^2 – statistic will be quite law. With a sample of T residuals, under the null hypothesis of no ARCH errors, the test statistic $LM = TR^2$ converges to χ^2 (q) distribution. If $LM = TR^2$ is sufficiently large, rejection of the null hypothesis that α_1 through α_q are jointly equal to zero is equivalent to rejecting the null hypothesis of no ARCH errors. On the other hand, if $LM = TR^2$ is sufficiently low, it is possible to conclude that there are no ARCH effects.

To obtain a better idea of actual process of fitting an ARCH model, let us reconsider the series of the Portuguese Imports used in the previous section. Recall that the Box-Jenkins approach led to estimate a model ARIMA(3,1,0) with the form (1). Diagnostic checks of residuals for this model did not indicate the presence of serial correlation, but there was a period of unusual volatility that could be characteristic of an ARCH process. Now, the aim is to examine the autocorrelation function of the squared residuals to find out the order of AR(q) model for them, which is equivalent to ARCH (q) model.

As it can be seen from Figure 4a, the null hypothesis of no ARCH process is rejected because two individual partial coefficients of autocorrelation are statistically significant for the 5 % level of significance.

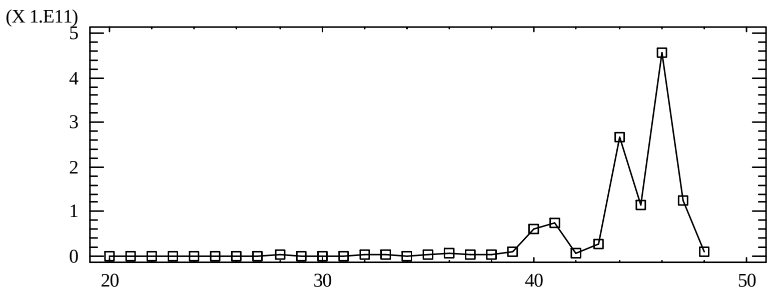


Figure 4 Squared residuals of the ARIMA(3,1,0) model

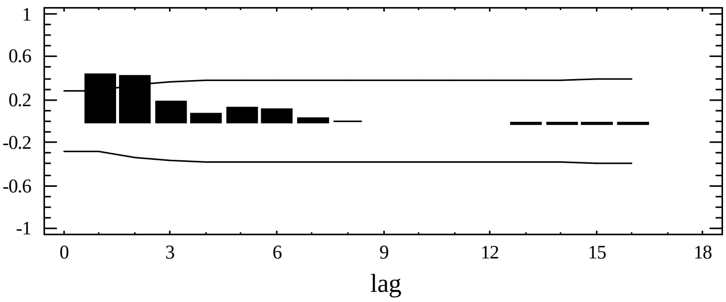


Figure 4a ACF of the squared residuals of ARIMA (3,1,0) model

We will use Lagrangian multiplier test to find out the order of the ARCH model. We perform the second order regression, of the following form

$$h_t = \hat{a}_t^2 = \alpha_0 + \alpha_1 \hat{a}_{t-1}^2 + \alpha_2 \hat{a}_{t-2}^2 + v_t ,$$

and its estimation is

$RESImpSQ_t$ = 1,12301E10 + 0,296469*$RESImpSQ_{t-1}$ + 0,303303*$RESImpSQ_{t-2}$,

(1.11E10) (0,147) (0,149)

where h_t = $RESImpSQ_t$ is the name of squared residual of the model ARIMA(3,1,0)),

$RESImpSQ_{t-1} = \hat{a}_{t-1}^2$ and $RESImpSQ_{t-2} = \hat{a}_{t-2}^2$.

All coefficients of the estimated regression model are statistically significant at 5 % level of significance, but not constant (standard errors in parentheses). Further statistics of regression are coefficient of determination R^2 = 0,255761, standard error of estimation = 7,07716E10 and Durbin-Watson statistic = 1,83. The value of the Langrangian multiplier is

$$LM = T R^2 = 48 * 0,255761 = 12,77$$

Since LM > $\chi_{0,05}^2(2) = 5{,}99$, we can reject null hypothesis and conclude that an ARCH(2) model is appropriate for modelling volatility in errors of the Import series.

The same results could be obtained by means of Ljung-Box Q-statistics used for the first 4 autocorrelation coefficients of the squared residuals.

The test statistic $Q = T(T+2)\sum_{k=1}^{m} r_k /(T-k) =$ 48*50 [(0,43/47) +(0,42/46) + (0,19/45) + (0,07/44)] = 57,82 and because it is greater than $\chi_{0,05}^2(4) = 9{,}49$ we conclude again, that there is an ARCH(2) effect.

3. Conclusion

Forecasts made by ARIMA (3,1,0) model assume time constant standard error of forecasts with value SE = 163 940, whereas forecasts made by ARIMA (3,1,0) model together with ARCH(2) model assume that the variance is a geometrically declining weighted average of the variance in the previous two years. This means that, for the future value of the Portuguese imports in 1996, we could expect smaller value for the forecast standard error (SE = 32 825). Hence, the Portuguese imports predictions of the two models should be similar, but the confidence intervals surrounding the forecasts will differ.

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