The Estimation of Number of Bootstrap Replications for the Parametric Bootstrap

Jana Kubanová
Faculty of Economics and Administration, University of Pardubice

Abstract

The problem of number of parametric bootstrap replications that are necessary to realize at bias estimation is solved in this paper. It shows results of simulations demonstrating extent of error that we can cause at different number of bootstrap replications.

There are many methods in statistics that solve how to cope with the problem of small number of statistical data. All these methods have common lack. Although each of them can react to the same initial conditions in a different way but generally all of these methods are characterized by low reliability of their results.

That is reason why new methods, that solve this problem in the most general way, are explored. The bootstrap methods belong to relatively new methods [3], whose practical use started with a mass introduction of computer technology. The basic idea of bootstrap method is generation of a great number of new samples on the base of information obtained from original sample, but its size is insufficient for statistical inferences.

In principle there are two methods of bootstrap – parametric and nonparametric [3]. Parametric bootstrap is used in this paper, where the resampled samples are realized by generating values from distribution $\hat{F}$, that is estimate of a real population with distribution F. Differences between $\hat{F}$ and F in parametric bootstrap consist in the fact that parameters $\boldsymbol{\Theta}$ of distribution F are in $\hat{F}$ replaced with their estimates $\hat{\Theta}$.

Our target is to examine the accuracy of bootstrap estimation in dependence on number of bootstrap replications. We will deal with properties of bias of sample average in the concrete way. In order to evaluate the results obtained by simulations, the basic sample has to be generated from some known distribution.

Let's assume that $x_1, x_2, \ldots, x_n$ is some realization of random sample from $N(\mu, \sigma)$ distribution. When parameter μ is estimated with sample average $\bar{x}$ and parameter σ with sample standard deviation $s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$, then the parametric bootstrap replication of sample $x_1, x_2, \ldots, x_n$ is random sample $x_1^*, x_2^*, \ldots, x_n^*$, that is from $N(\bar{x}, s)$ distribution. We obtain R realizations of bootstrap samples after R bootstrap replications.

Let's indicate bootstrap average $\overline{X}^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*$, where $X_1^*, X_2^*, \ldots, X_n^*$ is random sample from $N(\bar{x}, s)$ distribution. Random variable $\overline{X}^*$ is normally distributed with mean $\bar{x}$ and standard deviation $\frac{s}{\sqrt{n}}$. Bootstrap estimate of bias of sample average $\overline{X}^*$ is the statistics

$B_R = \frac{1}{R}\sum_{r=1}^{R} \overline{X}_r^* - \bar{x}$ [1].

Random variable $\frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^*$ is approximately normally distributed with mean $\overline{x}$ and standard deviation $\sqrt{\frac{s^2}{Rn}}$. Then the bootstrap bias $\frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^* - \overline{x}$ is approximately normally distributed with mean 0 and standard deviation $\sqrt{\frac{s^2}{Rn}}$ as well.

We are interested in bootstrap bias distribution and mean of bias $E^*(B_R)$ distribution, also B_R - $E^*(B_R)$ in the following step. This bootstrap bias is obtained from R bootstrap replications. But $E^*(B_R) = 0$ in our case. Therefore $B_R - E^*(B_R) = B_R = \frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^* - \overline{x}$.

Random variable B_R is approximately normally distributed with mean 0 and standard deviation $\sqrt{\frac{s^2}{Rn}}$. Then random variable B_R - $E^*(B_R)$ is also approximately normally distributed with mean 0 and standard deviation $\sqrt{\frac{s^2}{Rn}}$. This random variable exactly expresses the estimation of an error that is made after finite number of bootstrap replications. After normal transformation $\frac{B_R}{s}\sqrt{Rn}$ we obtain normally distributed random variable with mean 0 and standard deviation 1.
Interval estimate of variable $E^*(B_R)$ is then

$$B_R + z_\alpha \frac{s}{\sqrt{Rn}} < E^*(B_R) < B_R + z_{1-\alpha} \frac{s}{\sqrt{Rn}},$$

where z_α is quantile of $N(0,1)$ distribution. $z_\alpha = \Phi^{-1}\left(\frac{\alpha}{2}\right)$.

We can estimate by means of this relation extent of the error that we can cause at given number R of bootstrap replications, at given extent of n of original random sample and at chosen value of α.

We used two normally distributed random samples, the first one with mean 100, standard deviation 10 and size 15 and the second one with mean 0, standard deviation 1 and size 15 to illustrate solved problem. 10 000 bootstrap replications were made for each of above samples and basic statistical characteristics were computed. These characteristics are presented in table 1.

Tab.1

parameter	***N(100,10) distribution***	***N(0,1) distribution***
$\overline{x}$	101,267	0,1007
s^2	117,662	1,0614
$\overline{x}_R^*$	101,238	0,0982
s_R^{2*}	109,947	0,9971

Figures 1 and 2 show the results from several simulations, changes of bias with increasing number of bootstrap replications at sampling from $N(100,10)$ and $N(0,1)$ distributions.

The problem is demonstrated at 5 repetitions of 2000 replications. Empirical biases were calculated for each value of R. We can note how the variability decreases as the simulation size increases and how the simulated values converge to the exact value. To answer the question, how large a value of R is needed, figures 1 and 2 suggest that R = 300 bootstrap replications could be adequate. Values of bias don't markedly change at larger value of R .

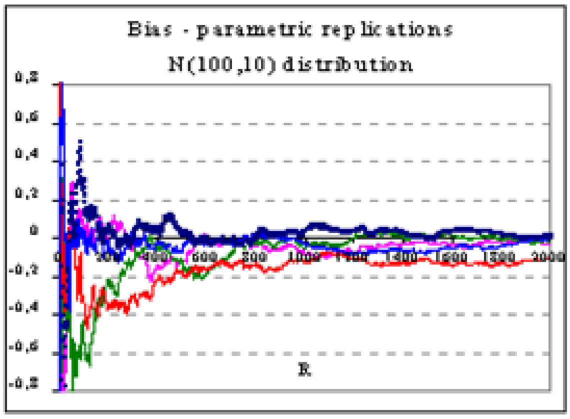

Fig.1

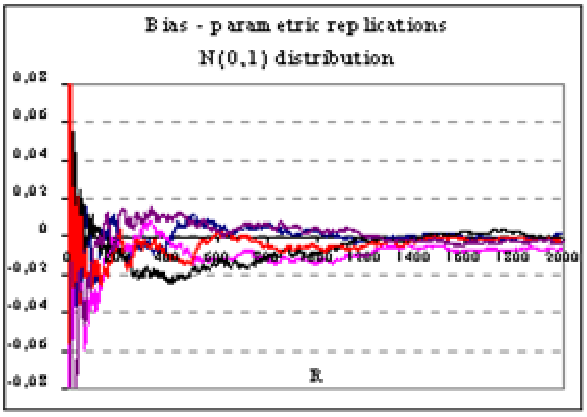

Fig.2

Let's simulate Q repetitions of R bootstrap replications. Bias estimates are calculated for each of Q repetitions and we calculate sample average of absolute values of these biases. We obtain variable $\overline{B}_{RQ} = \frac{1}{Q}\sum_{q=1}^{Q}\left|\frac{1}{R}\sum_{r=1}^{R}\overline{X}_r^* - \overline{x}\right|$ that can be considered to be measure of bias estimate error. Results of these simulations are presented in figures 3 and 4.

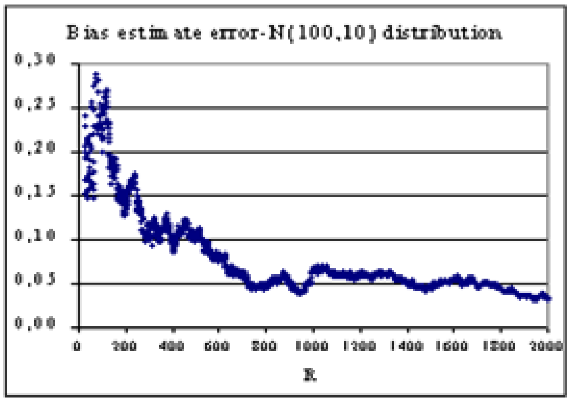

Fig.3

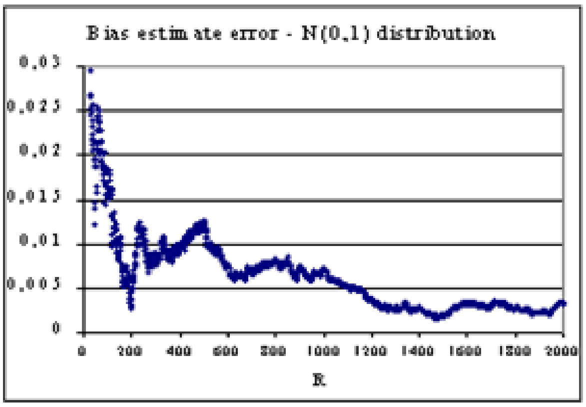

Fig.4

Figures 3 and 4 show that bias estimate error for 600 bootstraps replications from N(100,10) distribution was approximately 0,05, for 600 bootstraps replications from N(0,1) distribution it was approximately 0,007 and it was smaller than 0,005 for R >1100. Next the error didn't markedly change in both cases.

Figures 5 and 6 illustrate values of variance of bias estimates from simulated bootstrap samples (scatter diagram) and values of variance of bootstrap bias, calculated from $D^*(B_R) = \frac{s^2}{nR}$ (continuous curve). When R = 2000, these values don't differ significantly.

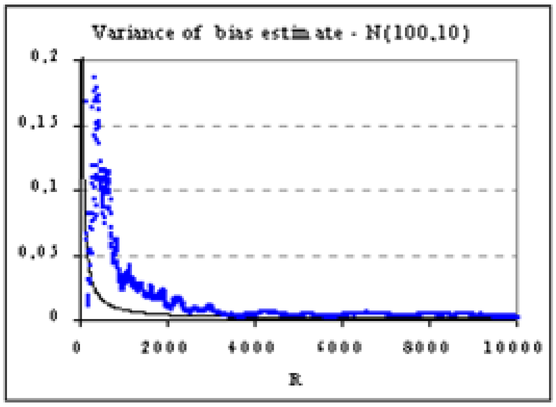

Fig.5

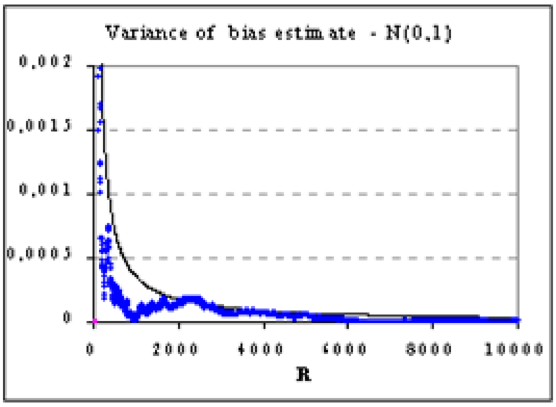

Fig.6

Next two figures 7 and 8 demonstrate absolute values of differences between the bias variance obtained after R replications of original realization of random sample and theoretical value obtained by calculation. Its value is $\frac{s^2}{nR}$. This difference is called the error of estimate of bias variance. Figures 7 and 8 suggest that this error is smaller than 0,01 for $R > 600$ at simulations from N(100,10) distribution and smaller than 0,0001 at N(0,1) distribution.

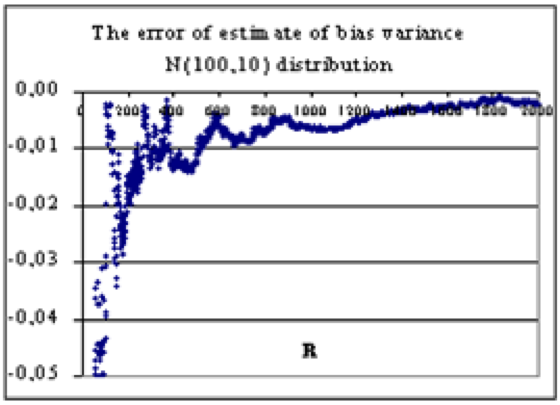

Fig.7

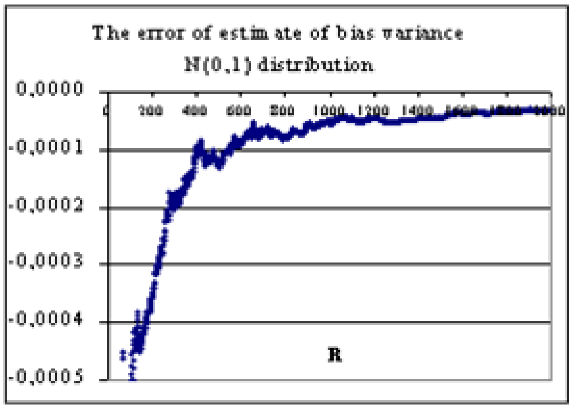

Fig.8

Tab. 2 Confidence interval for bias mean E^*B_R

N(100,10) distribution

(LL,UL are lower,upper limits of the confidence intervals)

R	***bias***	***LL***	***UL***
100	-0,379	-0,928	0,170
200	-0,131	-0,519	0,257
300	-0,085	-0,402	0,232
400	-0,039	-0,313	0,236
500	-0,130	-0,376	0,115
600	-0,184	-0,408	0,040
700	-0,125	-0,333	0,082
800	-0,119	-0,313	0,075
900	-0,129	-0,312	0,054
1000	-0,110	-0,284	0,063

R	***bias***	***LL***	***UL***
1000	-0,110	-0,284	0,063
2000	-0,083	-0,206	0,040
3000	-0,082	-0,182	0,018
4000	-0,037	-0,124	0,049
5000	-0,050	-0,128	0,027
6000	-0,046	-0,117	0,025
7000	-0,039	-0,105	0,026
8000	-0,039	-0,100	0,023
9000	-0,038	-0,095	0,020
10000	-0,029	-0,084	0,026

Table 2 shows 1 - α - percentage confidence interval [4], for the bias mean E^*B_R for 100 –1 000 bootstrap replications (left part) and for 1000 – 10000 bootstrap replications (right part) from *N*(100,10) distribution and for α = 0,05. The results of calculations are expressed in a graphic way in the figures 9 and 10.

Limits of 95% confidence intervals for E^*B_R after 100 – 1 000 bootstrap replications from *N*(100,10) distribution are presented in the figure 9, in the figure 10 is the parallel confidence interval from 1000 till 10 000 replications. Two figures were constructed by the reason of different *y* scale and better lucidity.
Figures 9 and 10 suggest, that in case of simulation from *N*(100,10) distribution, the error of bias estimate is smaller than 0,3 when $R > 300$, smaller than 0,2 when $R > 750$ and smaller than 0,1 when $R > 3000$. When $R = 10\,000$, the error is 0,055.

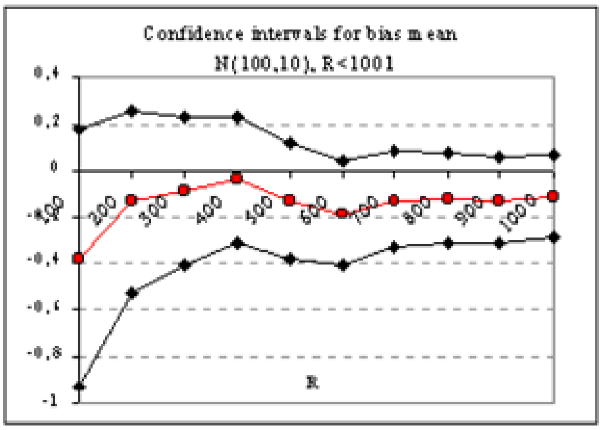

Fig.9

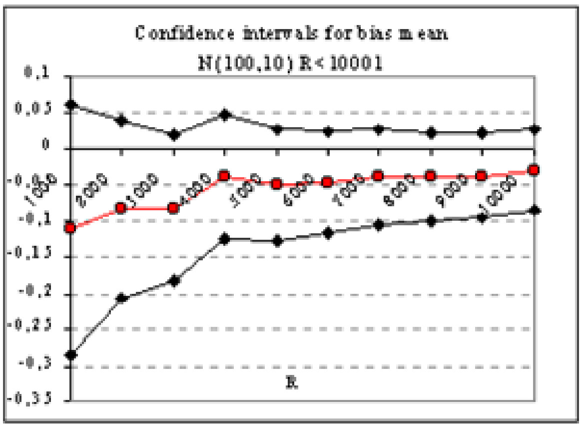

Fig.10

Tab. 3 Confidence interval for bias mean E^*B_R
N(0,1) distribution
(*LL,UL* are lower, upper limits of the confidence intervals)

R	bias	LL	UL
100	-0,0209	-0,0730	0,0312
200	-0,0120	-0,0489	0,0249
300	-0,0162	-0,0463	0,0139
400	-0,0161	-0,0422	0,0099
500	-0,0138	-0,0371	0,0095
600	-0,0092	-0,0305	0,0121
700	-0,0064	-0,0261	0,0133
800	-0,0030	-0,0215	0,0154
900	-0,0021	-0,0195	0,0153
1000	-0,0010	-0,0175	0,0155

R	bias	LL	UL
1000	-0,0010	-0,0175	0,0155
2000	-0,0053	-0,0169	0,0064
3000	-0,0027	-0,0122	0,0068
4000	-0,0033	-0,0115	0,0050
5000	-0,0042	-0,0116	0,0031
6000	-0,0037	-0,0104	0,0030
7000	-0,0027	-0,0090	0,0035
8000	-0,0022	-0,0080	0,0037
9000	-0,0018	-0,0073	0,0037
10000	-0,0025	-0,0077	0,0027

Limits of 95% confidence intervals for E^*B_R after 100 – 1 000 bootstrap replications from *N*(0,1) distribution are presented in the figure 11, in the figure 12 is the parallel confidence interval from 1000 till 10 000 replications. Analogically to previous case, two figures were created.

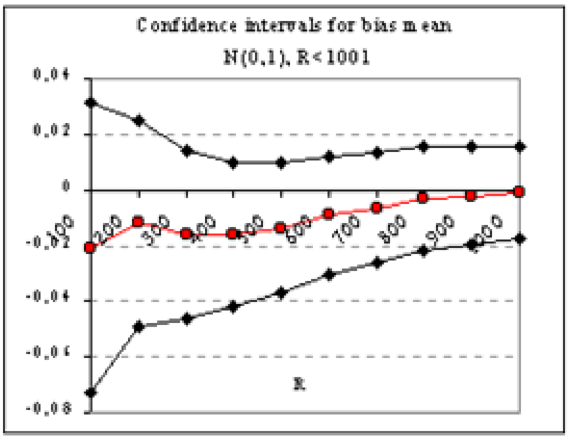

Fig.11

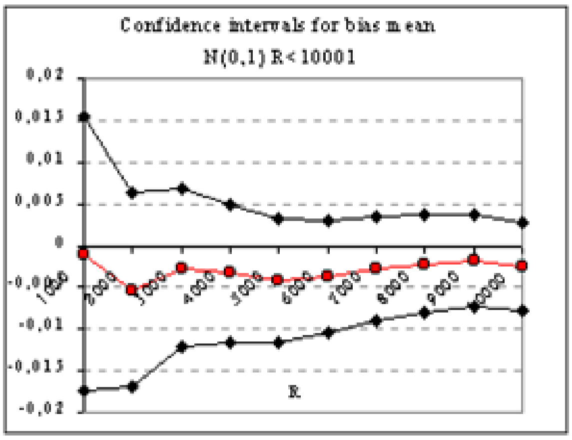

Fig.12

Figures 11 and 12 suggest, that in case of simulation from *N*(0,1) distribution, the error of bias estimate is smaller than 0,03 when R > 300, smaller than 0,02 when R > 700 and smaller than 0,01 when R > 2800. When R = 10 000, the error is 0,0052.

Finally it is possible to say that when the presumptions of normal distribution formulated in preliminary part of this paper were fulfilled, then we found out that 300 bootstrap replications were sufficient for the estimate of bias mean. Properties of estimated parameters weren't significantly improved when more bootstrap replications were made.

We tried to make bootstrap simulations for different values of μ and σ. The results obtained in these cases verified that 300 bootstrap replications are adequate for good estimate of bias mean.

Literature:

[1.] Davison A.C., Hinkley D.V. Bootstrap Methods and their Application. Cambridge University Press, 1997.

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Contacts address:
doc. PaedDr. Jana Kubanová, CSc.
University of Pardubice
Faculty of Economics and Administration
Department of Mathematics
53210 Pardubice
e-mail: jana.kubanova@upce.cz
00420 466036046

Review:
prof. RNDr. Otakar Prachař, CSc.
University of Pardubice
Faculty of Economics and Administration
Department of Mathematics